IX. Inhomogeneous equations: undetermined coefficients

Lesson Overview

- To solve the (linear, second-order, inhomogeneous, constant coefficient) differential equation

\[ ay'' + by' + cy = g(t) \]

first solve the homogeneous equation

\[ ay'' + by' + cy = 0 \]

by the methods of the previous lecture.

- Then find a particular solution to the inhomogeneous equation

\[ ay'' + by' + cy = g(t) \]

using undetermined coefficients. This means you guess something that looks like \( g(t) \), but has generic coefficients. Then you plug it in and solve for the coefficients.

<table>
<thead>
<tr>
<th>( g(t) )</th>
<th>Guess for ( y_{\text{par}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ke^t )</td>
<td>( Ae^t )</td>
</tr>
<tr>
<td>polynomial</td>
<td>( At^2 + Bt + C ) (same degree as ( g(t) ))</td>
</tr>
<tr>
<td>( k \sin 5t )</td>
<td>( A \sin 5t + B \cos 5t )</td>
</tr>
<tr>
<td>Combinations above.</td>
<td>Combinations above.</td>
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</tbody>
</table>

- If any term of your guess for \( y_{\text{par}} \) looks like any term of \( y_{\text{hom}} \), then multiply your whole guess by \( t \).

- Solve for constants after finding \( y_{\text{par}} \).
• If \( g(t) = \ln t, \tan t, \) etc., then abandon undetermined coefficients. Use variation of parameters (next lecture).

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**Example I**

Find the general solution to the differential equation:

\[
y'' - 3y' + 2y = 4e^{3t}
\]

\( r = 1, 2 \implies y_{\text{hom}} = c_1 e^t + c_2 e^{2t} \)

**Guess:** \( y_{\text{par}} = (\text{something that “looks like” } g(t)) = Ae^{3t}. \) \( (A \) to be determined.\) Plug in:

\[
\begin{align*}
y'_{\text{par}} &= 3Ae^{3t} \\
y''_{\text{par}} &= 9Ae^{3t}
\end{align*}
\]

\[
9Ae^{3t} - 3(3Ae^{3t}) + 2(Ae^{3t}) = 4Ae^{3t}
\]

\[
A = 2
\]

**General Solution:** \( y_{\text{gen}} = c_1 e^t + c_2 e^{2t} + 2e^{3t}. \)

Use IC to solve for constants after finding \( y_{\text{par}}. \)

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**Example II**

Find the general solution to the differential equation:

\[
y'' - 3y' + 2y = 4t^2
\]

\( r = 1, 2 \implies y_{\text{hom}} = c_1 e^t + c_2 e^{2t} \)

**Guess:** \( y_{\text{par}} = (\text{something that “looks like” } g(t)) \)
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\[ y'' - 3y' + 2y = 2A - 6At - 3B + 2At^2 + 2Bt + 2C = 4t^2 \]

Plug in:

\[
\begin{align*}
y_{\text{par}} & = At^2 + Bt + C \\
y'_{\text{par}} & = 2At + B \\
y''_{\text{par}} & = 2A \\
y'' - 3y' + 2y & = 2A - 6At - 3B + 2At^2 + 2Bt + 2C = 4t^2 \\
t^2 : & \quad 2A = 4 \quad \implies A = 2 \\
t : & \quad 2B - 6A = 0 \quad \implies B = 6 \\
\text{const} : & \quad 2A - 3B + 2C = 0 \quad \implies C = 7 \\
y_{\text{par}} & = 2t^2 + 6t + 7 \\
y_{\text{gen}} & = c_1e^t + c_2e^{2t} + 2t^2 + 6t + 7
\]

Example III

Find the general solution to the differential equation:

\[ y'' - 3y' + 2y = 5 \cos 2t \]

\[ r = 1, 2 \implies y_{\text{hom}} = c_1e^{t} + c_2e^{2t} \]

**Guess:** \( y_{\text{par}} = \) (something that “looks like” \( g(t) \))
Will Murray’s Differential Equations, IX. Inhomogeneous equations: undetermined coefficients

\[ y'' - 3y' + 2y = A \cos 2t + B \sin 2t \]  Plug in:
\[
\begin{align*}
y_{par} &= A \cos 2t + B \sin 2t \\
y'_{par} &= -2A \sin 2t + 2B \cos 2t \\
y''_{par} &= -4A \cos 2t - 4B \sin 2t \\
y'' - 3y' + 2y &= -4A \cos 2t - 4B \sin 2t - 3(-2A \sin 2t + 2B \cos 2t) + 2(A \cos 2t + B \sin 2t) \\
&= (-2A - 6B) \cos 2t + (6A - 2B) \sin 2t = 5 \cos 2t
\end{align*}
\]
\[-2A - 6B = 5
\]
\[6A - 2B = 0 \implies B = 3A
\]
\[-2A - 6(3A) = 5 \implies -20A = 5 \implies A = -\frac{1}{4} \implies B = -\frac{3}{4}
\]
\[
y_{par} = -\frac{1}{4} \cos 2t - \frac{3}{4} \sin 2t
\]
\[
y_{gen} = c_1 e^t + c_2 e^{2t} - \frac{1}{4} \cos 2t - \frac{3}{4} \sin 2t
\]

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**Example IV**

Find the general solution to the differential equation:
\[ y'' - 3y' + 2y = 5e^t \]
\[ r = 1, 2 \implies y_{hom} = c_1 e^t + c_2 e^{2t} \]

**Guess:** \( y_{par} = Ae^t \). This is doomed to fail, because this \( y_{par} \) is a copy of \( y_{hom} \)!

Use \( y_{par} = Ate^t \) instead.
\[
\begin{align*}
y'_{par} &= Ate^t + Ae^t \\
y''_{par} &= Ate^t + Ae^t + Ae^t \\
&= Ate^t + 2Ae^t \\
Ate^t + 2Ae^t - 3\left( Ate^t + Ae^t \right) + 2Ate^t &= 5e^t \quad \{t \text{ terms cancel.} \} \\
-Ae^t &= 5e^t \\
A &= -5 \\
y_{par} &= -5te^t \\
y_{gen} &= c_1 e^t + c_2 e^{2t} - 5te^t
\end{align*}
\]
Example V

Give an appropriate form for the particular solution to the differential equation:

\[ y'' - 3y' + 2y = t^4 + 7e^{3t} \]

\[ r = 1, 2 \implies y_{\text{hom}} = c_1 e^t + c_2 e^{2t} \]

**Strategy:** Solve \( L[y_{p_1}] = t^4 \) by guessing \( y_{p_1} := At^4 + Bt^3 + Ct^2 + Dt + E \). Then solve \( L[y_{p_2}] = 7e^{3t} \) by guessing \( y_{p_2} := Fe^{3t} \). Then let \( y_p := y_{p_1} + y_{p_2} \), so \( L[y_p] = t^4 + e^{3t} \).