Lesson Overview

- Euler’s method is a way to find numerical approximations for initial value problems that we can’t solve analytically.
- It is based on drawing lines along slopes in a direction field.

Formulas for Euler’s method

- Start with an initial value problem in the form \( y'(t) = f(t, y), y(t_0) = y_0 \).
- Choose a step size \( h \) (usually given).
- Start at \((t_0, y_0)\) and make iterative steps:

\[
\begin{align*}
  t_{n+1} & := t_n + h \\
y_{n+1} & = y_n + hf(t_n, y_n)
\end{align*}
\]

- Continue until you arrive at the value of \( t \) for which you need to approximate \( y(t) \).

Example I

Use Euler’s method with step size \( h = 0.1 \) to estimate \( y(0.4) \) in the initial value problem \( y' = 1 + t - y, y(0) = 1 \).
Example II

Solve the initial value problem

\[ y' = 1 + t - y, \quad y(0) = 1 \]

analytically. Compute \( y(0.4) \) and compare the answer with the result given by Euler’s method above.

\[
\begin{align*}
y' + y &= 1 + t \quad \{I(t) = e^t\} \\
y'e^t + ye^t &= e^t + te^t \\
(ye^t)' &= e^t + te^t \\
ye^t &= te^t + C \\
y &= t + Ce^{-t} \\
1 &= 0 + C \\
C &= 1 \\
y &= t + e^{-t} \\
y(0.4) &= 0.4 + e^{-0.4} \approx 1.07032 \\
\text{Euler: } y(0.4) &\approx 1.0561
\end{align*}
\]
We were off by about 0.014. That’s not bad.

Example III
Use Euler’s method with step size $h = 0.1$ to estimate $y(0.4)$ in the initial value problem $y' = t^2 + y^2, y(0) = 1$.

\[
\begin{align*}
(0, 1) &\rightarrow y' = 1 \\
(0.1, 1.1) &\rightarrow y' = 0.01 + 1.21 = 1.22 \\
(0.2, 1.222) &\rightarrow y' = 0.04 + 1.49328 = 1.53328 \\
(0.3, 1.37533) &\rightarrow y' = 0.09 + 1.89153 = 1.98153 \\
(0.4, 1.573) &\rightarrow y' = 0.09 + 1.89153 = 1.98153
\end{align*}
\]

So $y(0.4) \approx 1.573$.

Example IV
Use Euler’s method with step size $h = 0.3$ to estimate $y(0.6)$ in the initial value problem $y' = t - y, y(0) = 1$.

\[
\begin{align*}
(0, 1) &\Rightarrow y' = -1 \\
(0.3, 0.7) &\Rightarrow y' = 0.3 - 0.7 = -0.4 \\
(0.6, 0.58) &\Rightarrow y' = 0.3 - 0.7 = -0.4
\end{align*}
\]

So $y(0.4) \approx 0.58$.

Example V
Solve the initial value problem
\[y' = t - y, y(0) = 1\]
analytically. Compute $y(0.6)$ and compare the answer with the result given by Euler’s method above.

\[
\begin{align*}
y' + y &= t \quad \{I(t) = e^t\} \\
y' e^t + ye^t &= te^t \\
(ye^t)' &= te^t \\
ye^t &= te^t - e^t + C \\
y &= t - 1 + Ce^{-t} \\
y(0) &= 1 \quad 1 = -1 + C \\
C &= 2 \\
y &= t - 1 + 2e^{-t} \\
y(0.6) &= 0.6 - 1 + 2e^{-0.6} \approx 0.697623 \\
\text{Euler: } y(0.6) &\approx 0.58
\end{align*}
\]