XI. Review of power series

Lesson Overview

- A function \( f(x) \) has a Taylor Series expansion around a point \( x_0 \):

\[
TS(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \text{ where } a_n = \frac{f^{(n)}(x_0)}{n!}
\]

If \( x_0 = 0 \), it’s also called Maclaurin Series.

- Common Maclaurin Series to remember from calculus:

\[
\begin{align*}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots
\end{align*}
\]

\[
\begin{align*}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \text{ for } |x| < 1
\end{align*}
\]
Any power series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) has a radius of convergence \( R \) around \( x_0 \), i.e. when you plug in values for \( x \) satisfying \( |x - x_0| < R \), it converges. It might or might not converge at the endpoints \( x = x_0 - R, x = x_0 + R \).

**Extreme cases:**

- \( R = 0 \). It only converges for \( x = x_0 \).
- \( R = \infty \). Then it converges for all \( x \in \mathbb{R} \).
- To find the radius of convergence, we usually use the **Ratio Test**: A series converges if
  \[
  \lim_{n \to \infty} \left| \frac{\text{term}_{n+1}}{\text{term}_n} \right| < 1.
  \]

- We must check the endpoints separately (using a **non-Ratio** test).

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**Example I**

Identify the following power series as an elementary function:

\[
1 + 3x^2 + \frac{9}{2} x^4 + \frac{9}{2} x^6 + \frac{27}{8} x^8 + \cdots
\]

\[
= 1 + 3x^2 + \frac{9}{2} x^4 + \frac{27}{6} x^6 + \frac{81}{24} x^8 + \cdots
\]

\[
= 1 + 3x^2 + \frac{(3x^2)^2}{2!} + \frac{(3x^2)^3}{6} + \frac{(3x^2)^4}{24} + \cdots
\]

\[
= e^{3x^2}
\]
Example II

Find the Maclaurin Series for \( f(x) = \ln(1 - x) \).

**Lesson from Calc II:** Writing down \( f(x), f'(x), f''(x), \ldots \) is usually the worst way to find a Taylor Series.

Instead, note that

\[
\ln(1 - x) = -\int \frac{dx}{1 - x} = -\int (1 + x + x^2 + x^3 + \cdots) \, dx = C - x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots
\]

Plug in \( x = 0 \) to get \( C = 0 \).

So \( a_n = 0 \) for \( n = 0 \), \( a_n = -\frac{1}{n} \) for \( n = 1, 2, 3, \ldots \).

\[
\ln(1 - x) = \sum_{n=1}^{\infty} \left( -\frac{x^n}{n} \right)
\]

Example III

Find the interval of convergence for the Maclaurin Series for \( \ln(1 - x) \).

Use the Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{-x^{n+1}}{n+1} \right| = |x| < 1
\]

So it converges for \( |x| < 1 \). Ratio \( R = 1 \) around \( x_0 = 0 \).

Check the endpoints separately (using a non-Ratio test): \( x = 1 \) gives \(-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots\),
which diverges (Harmonic Series/$p$-series).

$x = -1$ gives $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, which converges (Alternating Series Test).

So this series converges for $[-1 \leq x < 1]$, or $[-1, 1)$. This was predictable since $\ln(1 - x)$ blows up at $x = 1$.

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Example IV

Use power series to solve the following integral:

$$\int e^{x^2} \, dx$$

$\int e^{x^2} \, dx$ can not be done by any integration technique you learned in Calc II (substitution, parts, partial fractions, etc.). That’s because there is no “elementary function” whose derivative is $e^{x^2}$. But we can find a series that works:

$$\int e^{x^2} \, dx = \int \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) \, dx$$

$$= C + x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \cdots$$

Alternately,

$$\int \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right) \, dx = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}.$$

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Example V

Suppose $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Find power series expressions for $y'(x)$ and $y''(x)$ and shift the indices of summation so that they start at $n = 0$.  

\[ y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ } \begin{cases} \text{Omit the } n = 0 \text{ term} \\ \text{because it is 0 anyway.} \end{cases} \]
\[ = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \text{ } \begin{cases} \text{Shift the index of} \\ \text{summation by 1.} \end{cases} \]
\[ = \sum_{n=0}^{\infty} (n + 1)a_{n+1} x^{n+1} \text{ } \begin{cases} \text{Mnemonic: If you lower} \\ \text{the } n \text{ in the index by 1, then} \\ \text{raise the } n \text{'s in the formula} \text{ by 1.} \end{cases} \]

\[ y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \text{ } \begin{cases} \text{Omit the } n = 0 \text{ and } n = 1 \text{ terms because they are 0.} \end{cases} \]
\[ = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2} x^n \text{ } \begin{cases} \text{Shifting } n \text{ by 2.} \end{cases} \]