I. Linear equations

Lesson Overview

• Today we’ll learn how to solve linear differential equations:

\[ y'(x) + P(x)y(x) = Q(x) \]

• Notes on the form:

1. Linear means that we think of \( y \) and \( y' \) as the variables (not \( x \) and \( y \)). We think of as \( P(x) \) and \( Q(x) \) as coefficients: The equation has the form \( y' + Py = Q \), which would be a line.

2. If there is a coefficient in front of \( y'(x) \), make sure you divide it away before using the algorithm below.

How to solve linear equations

\[ y'(x) + P(x)y(x) = Q(x) \]

1. Calculate the integrating factor

\[ I(x) := e^{\int P(x) \, dx} \]

and multiply that by both sides.

2. This makes the left hand side into

\[ e^{\int P(x) \, dx} y' + P(x) e^{\int P(x) \, dx} y = Iy' + I'y = (Iy)', \]

so we can then integrate both sides.
Solving linear equations

\[ y'(x) + P(x)y(x) = Q(x) \]

3. Then you’ll get

\[ I(x)y(x) = \int I(x)Q(x) \, dx + C \]

and you can solve for \( y(x) \).

Further notes

- If \( P(x) \) is negative, make sure to include that in finding \( I(x) \). And remember that \( e^{-\ln(\text{cucumber})} \) doesn’t simplify to \(-\text{cucumber})! It’s \( \frac{1}{\text{cucumber}} \).

- When you’re integrating \( P(x) \) to find the integrating factor \( I(x) \), it’s ok to leave off the constant \( C \).

- However, when you’re integrating both sides of the equation, the \( C \) is very important. And it’s important that you add it when you do the integration and keep track of it in the ensuing algebra.

Example I

Find the general solution to the following differential equation:

\[ y' + xy = x^3 \]
Example I

\[ y'(x) + xy = x^3 \]

Multiply both sides by \( e^{x^2} \):

\[ e^{x^2} y' + x e^{x^2} y = x^3 e^{x^2} \]

**Point:** The LHS is now \( (ye^{x^2})' \), using the Product Rule.

\[ (ye^{x^2})' = x^3 e^{x^2} \quad \{ \text{Integrate both sides:} \} \]

**RHS:**

\[ u := \frac{x^2}{2} \quad du = x \, dx \]

\[ \int x^3 e^{x^2} \, dx = \int x^2 e^{x^2} x \, dx = \int 2ue^u \, du \]

Use parts:

\[ = 2(ue^u - e^u) + C = 2 \left( \frac{x^2 e^{x^2}}{2} - e^{x^2} \right) + C = x^2 e^{x^2} - 2e^{x^2} + C \]

\[ ye^{x^2} = (x^2 - 2)e^{x^2} + C \]

\[ y = \frac{(x^2 - 2)e^{x^2} + C}{e^{x^2}} \quad \{ \text{Not } y = x^2 - 2 + C! \} \]

Now use IC (if given) to get \( C \).

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Example II

Solve the following initial value problem:

\[ (\cos x)y' + (\sin x)y = \cos^5 x \sin x, \quad y(0) = 2 \]
\[(\cos x)y' + (\sin x)y = \cos^5 x \sin x, \, y(0) = 2\]

\[
\begin{align*}
(\cos x)y' + (\sin x)y &= \cos^5 x \sin x, \, y(0) = 2 \\
y' + (\tan x)y &= \cos^4 x \sin x \\
I(x) &= e^{\int \tan x \, dx} \\
&= e^{-\ln \cos x} \\
&= \sec x \\
(\sec x)y' + (\sec x \tan x)y &= \cos^3 x \sin x \\
\end{align*}
\]

\[\text{At this point, check}^* \text{ whether the LHS really is the derivative of } (\sec x)y.\]

\[
\begin{align*}
(\sec x)y &= -\frac{1}{4} \cos^4 x + C \\
y &= C \cos x - \frac{1}{4} \cos^5 x \\
y(0) &= C - \frac{1}{4} = 2 \\
C &= \frac{9}{4} \\
y &= \frac{9}{4} \cos x - \frac{1}{4} \cos^5 x
\end{align*}
\]

Example III

For the linear differential equation

\[(x + 1)y' - y = \sin x, \]

what is \(I(x)\)?

\[\frac{1}{x + 1}\]

Example IV
For the linear differential equation

\[(\sin x)y' - (\cos x)y = x^2,\]

what is \(I(x)\)?

\[\csc x\]

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**Example V**

Solve the initial value problem:

\[(t + 1)y' - 3y = t, \quad y(1) = 2\]

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**Example V**

\[(t + 1)y' - 3y = t, \quad y(1) = 2\]
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\[
y' - \frac{3}{t+1}y = \frac{t}{t+1}
\]

\[
I(t) := e^{-\int \frac{3}{t+1} dt} = e^{-3\ln(t+1)} = \frac{1}{(t+1)^3}
\]

\[
\frac{y'}{(t+1)^3} - \frac{3}{(t+1)^4}y = \frac{t}{(t+1)^4}
\]

\[
\left(\frac{y}{(t+1)^3}\right)' = \frac{t}{(t+1)^4}
\]

\[
y\frac{y}{(t+1)^3} = \int \frac{t}{(t+1)^4} dt \quad \{u := t + 1, du = dt\}
\]

\[
= \int \left(\frac{u-1}{u^4}\right) du
\]

\[
= \int \left(\frac{1}{u^3} - \frac{1}{u^4}\right) du
\]

\[
= -\frac{1}{2(t+1)^2} + \frac{1}{3(t+1)^3} + C
\]

\[
y = -\frac{1}{2}(t+1) + \frac{1}{3} + C(t+1)^3
\]

\[
2 = -\frac{1}{2}(2) + \frac{1}{3} + 8C
\]

\[
\frac{8}{3} = 8C
\]

\[
C = \frac{1}{3}
\]

\[
y = \frac{1}{3} \left[(t+1)^3 + 1\right] - \frac{1}{2}(t + 1)
\]

\[
= \frac{1}{3} t^3 + t^2 + \frac{1}{2} t + \frac{1}{6}
\]

Example VI
Find the general solution to the following
differential equation:

\[ xy' + 3y = \cos x, \quad x > 0 \]

Example VI

\[ y' + \frac{3}{x} y = \frac{\cos x}{x} \]

\[ I(x) = e^{\int \frac{3}{x} \, dx} = e^{3 \ln x} = e^3 = x^3 \]

\[ x^3 y' + 3x^2 y = x^2 \cos x \]

\[
\begin{array}{c|c}
 x^2 & \cos x \\
 2x & \sin x \\
 2 & -\cos x \\
 0 & -\sin x \\
\end{array}
\]

\[ x^3 y = x^2 \sin x + 2x \cos x - 2 \sin x + C \]

\[ y = \frac{x^2 \sin x + 2x \cos x - 2 \sin x + C}{x^3} \]