XVI. Inverse Laplace transforms

Lesson Overview

- We are given the Laplace transform $L\{f\}$ of a function, and we try to reconstruct the original function $f(t)$.
- $L\{f\}$ will be a function of s, and we usually have to run partial fractions on it.
- Then we use the following table.

Common Laplace transforms

f	$L\{f\}$	f	$L\{f\}$	f	$L\{f\}$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$	$\cos bt$	$\frac{s}{s^2+b^2}$
t	$\frac{1}{s^2}$	te^{at}	$\frac{1}{(s-a)^2}$	$\sin bt$	$\frac{b}{s^2+b^2}$
t^2	$\frac{2}{s^3}$	$\vdots$		$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$

f	$L\{f\}$	f	$L\{f\}$
1	$\frac{1}{s}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
t	$\frac{1}{s^2}$	$\cos bt$	$\frac{s}{s^2+b^2}$
t^2	$\frac{2}{s^3}$	$\sin bt$	$\frac{b}{s^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
e^{at}	$\frac{1}{s-a}$	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
te^{at}	$\frac{1}{(s-a)^2}$		

Example I

Find the inverse Laplace transform of $\frac{7s+5}{s^2+s-2}$.

Partial fractions:

$$\begin{aligned}
\frac{7s+5}{s^2+s-2} &= \frac{A}{s+2}+\frac{B}{s-1} \\
7s+5 &= A(s-1)+B(s+2) \\
\text{Plug in } s=1: \quad 12 &= 3B \Longrightarrow B=4 \\
\text{Plug in } s=-2: \quad -9 &= -3A \Longrightarrow A=3 \\
\frac{7s+5}{s^2+s-2} &= \frac{3}{s+2}+\frac{4}{s-1} \\
\text{Look at the chart:} \quad L\left\{e^{at}\right\} &= \frac{1}{s-a} \\
f(t) &= \boxed{3e^{-2t}+4e^{t}}
\end{aligned}$$

Example II

Find the inverse Laplace transform of $\frac{-2s-3}{s^2+3s}$.

Partial fractions:

$$\begin{aligned}
\frac{-2s-3}{s^2+3s} &= \frac{A}{s} + \frac{B}{s+3} \\
-2s-3 &= A(s+3) + Bs \\
\text{Plug in } s=0: \quad -3 &= 3A \Longrightarrow A = -1 \\
\text{Plug in } s=-3: \quad 3 &= -3B \Longrightarrow B = -1 \\
\frac{-2s-3}{s^2+3s} &= -\frac{1}{s} - \frac{1}{s+3} \\
\text{Look at the chart:} \quad L\{1\} &= \frac{1}{s} \\
L\left\{e^{at}\right\} &= \frac{1}{s-a} \\
f(t) &= \boxed{-1-e^{-3t}}
\end{aligned}$$

Example III

Find the inverse Laplace transform of $\frac{s+4}{s^2+4s+5}$.

Complete the square:

$$\begin{aligned}
\frac{s+4}{s^2+4s+5} &= \frac{s+4}{s^2+4s+4+1} = \frac{s+4}{(s+2)^2+1} \\
\text{Look at the chart:} \quad L\left\{e^{at}\cos bt\right\} &= \frac{s-a}{(s-a)^2+b^2} \\
L\left\{e^{at}\sin bt\right\} &= \frac{b}{(s-a)^2+b^2} \\
\frac{s+4}{(s+2)^2+1} &= \frac{s+2}{(s+2)^2+1} + \frac{2}{(s+2)^2+1} \\
f(t) &= \boxed{e^{-2t}\cos t + 2e^{-2t}\sin t}
\end{aligned}$$

Example IV

Find the inverse Laplace transform of $\frac{4s^2-11s+11}{(s-1)^3}$.

Partial fractions:

$$\begin{aligned}
\frac{4s^2-11s+11}{(s-1)^3} &= \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3} \\
4s^2-11s+11 &= A(s-1)^2+B(s-1)+C = As^2-2As+A+Bs-B+C = \\
s^2\text{:}\quad A &= 4 \\
s\text{:}\quad B-2A &= -11 \Longrightarrow B=-3 \\
\text{constant:}\quad A-B+C &= 11 \Longrightarrow C=4 \\
\frac{4s^2-11s+11}{(s-1)^3} &= \frac{4}{s-1}-\frac{3}{(s-1)^2}+\frac{4}{(s-1)^3} \\
\text{Look at the chart:}\quad L\left\{e^{at}\right\} &= \frac{1}{s-a} \\
L\left\{te^{at}\right\} &= \frac{1}{(s-a)^2} \\
L\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} &\Longrightarrow L\left\{t^2e^{at}\right\} = \frac{2}{(s-a)^3} \\
f(t) &= \boxed{4e^t-3te^t+2t^2e^t}
\end{aligned}$$

Example V

Find the inverse Laplace transform of $\frac{4s^2+4s}{(s+2)(s^2+4)}$.

Partial fractions:

$$\begin{aligned}
\frac{4s^2+4s}{(s+2)(s^2+4)} &= \frac{A}{s+2}+\frac{Bs+C}{s^2+4} \\
4s^2+4s &= A(s^2+4)+(Bs+C)(s+2) \\
s=-2:\quad 8 &= 8A \Longrightarrow A=1 \\
4s^2+4s &= s^2+4+(Bs+C)(s+2) = s^2+4+Bs^2+Cs+2Bs+2C \\
s^2:\quad 4 &= 1+B \Longrightarrow B=3 \\
\text{constant:}\quad 0 &= 4+2C \Longrightarrow C=-2 \\
\frac{4s^2+4s}{(s+2)(s^2+4)} &= \frac{1}{s+2}+\frac{3s-2}{s^2+4} = \frac{3s}{s^2+4}-\frac{2}{s^2+4}
\end{aligned}$$

Look at the chart:

$$\begin{aligned}
L\left\{e^{at}\right\} &= \frac{1}{s-a} \\
L\{\cos bt\} &= \frac{s}{s^2+b^2} \\
L\{\sin bt\} &= \frac{b}{s^2+b^2} = \frac{2}{s^2+4} \text{ if } b=2 \\
f(t) &= \boxed{e^{-2t}+3\cos 2t-\sin 2t}
\end{aligned}$$