VII. Random Variables

Intuition

- A random variable Y is a quantity you keep track of during an experiment.
- **Example**: In the World Series, the Yankees play the Giants for 7 games. There are 2^7 possible outcomes, but we're only really interested in

$$Y := \text{number of games the Yankees win.}$$

This is different from the rules for the real World Series, in which one might play less than 7 games.

Intuition

- Sometimes it's useful to think of a payoff on an experiment.
- **Example**: You draw a card, and if it's ace through nine, I pay you that amount. If it's a ten or a face card, you pay me \$10.

$$Y := \text{amount of money you make}$$

Note that this Y could be positive or negative.

$$\begin{aligned} Y(\text{ace}) &:= 1 \\ Y(2) &:= 2 \\ &\vdots \\ Y(\text{jack}) &:= -10 \end{aligned}$$

Definition

- A random variable is a function from a sample space to $\mathbb{R}$, the set of real numbers.

$$Y : S \to \mathbb{R}$$

Example: In the World Series, how many games do the Yankees win?

$$\begin{aligned} Y(WWWLLWL) &= 4 \\ Y(LLLLLLW) &= 1 \\ Y(WLWLWLW) &= 4 \\ &\vdots \end{aligned}$$

Probability distributions

- $\underline{p(y) = P(Y = y)}$ is the sum of all the probabilities of the outcomes for which $Y = y$:

$$\underline{p(y) = P(Y = y)} := \sum_{E \in S, Y(E) = y} P(E)$$

- The function $p(y)$ is the <u>probability distribution</u> of the random variable Y.

Example I

You draw a card from a standard 52-card deck. If it's ace through nine, I pay you that amount. If it's a ten or a face card, you pay me \$10. What is the probability distribution for this random variable?

$$\begin{aligned}
p(0) &= \boxed{0} \\
p(1) = P(Y = 1) &= \frac{4}{52} = \boxed{\frac{1}{13}} \\
p(2) = P(Y = 1) &= \frac{4}{52} = \boxed{\frac{1}{13}} \\
&\vdots \\
p(9) = P(Y = 1) &= \frac{4}{52} = \boxed{\frac{1}{13}} \\
p(-10) = P(Y = -10) &= p(-10) = \frac{16}{52} = \boxed{\frac{4}{13}}
\end{aligned}$$

Example II

Flip a fair coin 10 times. Let Y be the number of heads. What is the probability distribution for this random variable?

$$p(y) = P(Y = y) = \boxed{\frac{\binom{10}{y}}{2^{10}}, 0 \leq y \leq 10}$$

Example III

Roll a die repeatedly until you get a 6. Let Y be the number of rolls. What is the probability distribution for this random variable?

$$\begin{aligned} p(1) = P(Y = 1) &= \frac{1}{6} \\ p(2) &= \frac{5}{6} \cdot \frac{1}{6} \\ p(3) &= \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \\ &\vdots \\ p(y) &= \boxed{\left(\frac{5}{6}\right)^{y-1} \cdot \frac{1}{6}, 1 \leq y < \infty} \end{aligned}$$

Example IV

Manchester United plays Liverpool FC for three matches. In any given match, Liverpool is twice as likely to win as Manchester. There are no ties. Let Y be the number of matches Liverpool wins. What is the probability distribution for this random variable?

In each match, Liverpool wins with probability $\frac{1}{3}$, Manchester with probability $\frac{2}{3}$.

$$
\begin{aligned}
p(0) = P(Y=0) &= P(LLL) = \left(\frac{1}{3}\right)^3 = \boxed{\frac{1}{27}} \\
p(1) &= P(WLL, LWL, LLW) = 3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = \boxed{\frac{2}{9}} \\
p(2) &= P(WWL, WLW, LLW) = 3\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \boxed{\frac{4}{9}} \\
p(3) &= P(WWW) = \left(\frac{2}{3}\right)^3 = \boxed{\frac{8}{27}}
\end{aligned}
$$

Example V

You and a friend each flip a coin. If both flips are heads, your friend pays you \$10; if both are tails, he pays you \$5. If the coins do not match, you pay him \$5. Let Y be the amount you win. What is the probability distribution for this random variable?

$$
\begin{aligned}
p(0) = P(Y=0) &= \boxed{0} \\
p(5) &= P(TT) = \boxed{\frac{1}{4}} \\
p(10) &= P(HH) = \boxed{\frac{1}{4}} \\
p(-5) &= P(HT, TH) = \boxed{\frac{1}{2}}
\end{aligned}
$$