VII. Random Variables

Intuition

• A random variable $Y$ is a quantity you keep track of during an experiment.

• **Example:** In the World Series, the Yankees play the Giants for 7 games. There are $2^7$ possible outcomes, but we’re only really interested in

  $$Y := \text{number of games the Yankees win.}$$

This is different from the rules for the real World Series, in which one might play less than 7 games.

Intuition

• Sometimes it’s useful to think of a payoff on an experiment.

• **Example:** You draw a card, and if it’s ace through nine, I pay you that amount. If it’s a ten or a face card, you pay me $10.

  $$Y := \text{amount of money you make}$$

Note that this $Y$ could be positive or negative.
\[ Y(\text{ace}) := 1 \]
\[ Y(2) := 2 \]
\[ \vdots \]
\[ Y(\text{jack}) := -10 \]

**Definition**

- A random variable is a function from a sample space to \( \mathbb{R} \), the set of real numbers.

\[ Y : S \to \mathbb{R} \]

**Example:** In the World Series, how many games do the Yankees win?

\[ Y(\text{WWWLLWL}) = 4 \]
\[ Y(\text{LLLLLLW}) = 1 \]
\[ Y(\text{WLWLWLL}) = 4 \]
\[ \vdots \]

**Probability distributions**

- \( p(y) = P(Y = y) \) is the sum of all the probabilities of the outcomes for which \( Y = y \):

\[
p(y) = P(Y = y) := \sum_{E \in S, Y(E) = y} P(E)\]
• The function $p(y)$ is the probability distribution of the random variable $Y$.

Example I
You draw a card from a standard 52-card deck. If it’s ace through nine, I pay you that amount. If it’s a ten or a face card, you pay me $10. What is the probability distribution for this random variable?

\[
p(0) = 0 \\
p(1) = P(Y = 1) = \frac{4}{52} = \frac{1}{13} \\
p(2) = P(Y = 1) = \frac{4}{52} = \frac{1}{13} \\
\vdots \\
p(9) = P(Y = 1) = \frac{4}{52} = \frac{1}{13} \\
p(-10) = P(Y = -10) = p(-10) = \frac{16}{52} = \frac{4}{13}
\]

Example II
Flip a fair coin 10 times. Let $Y$ be the number of heads. What is the probability distribution for this random variable?
$p(y) = P(Y = y) = \frac{\binom{10}{y}}{2^{10}}, 0 \leq y \leq 10$

**Example III**

Roll a die repeatedly until you get a 6. Let $Y$ be the number of rolls. What is the probability distribution for this random variable?

\[
p(1) = P(Y = 1) = \frac{1}{6} \\
p(2) = \frac{5}{6} \cdot \frac{1}{6} \\
p(3) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \\
\vdots \\
p(y) = \left(\frac{5}{6}\right)^{y-1} \cdot \frac{1}{6}, 1 \leq y < \infty
\]

**Example IV**

Manchester United plays Liverpool FC for three matches. In any given match, Liverpool is twice as likely to win as Manchester. There are no ties. Let $Y$ be the number of matches Liverpool wins. What is the probability distribution for this random variable?
In each match, Liverpool wins with probability \( \frac{1}{3} \),
Manchester with probability \( \frac{2}{3} \).

\[
p(0) = P(Y = 0) = P(LLL) = \left( \frac{1}{3} \right)^3 = \frac{1}{27}
\]

\[
p(1) = P(WLL, LWL, LLW) = 3 \left( \frac{2}{3} \right) \left( \frac{1}{3} \right)^2 = \frac{2}{9}
\]

\[
p(2) = P(WWL, WLW, LLW) = 3 \left( \frac{2}{3} \right)^2 \left( \frac{1}{3} \right) = \frac{4}{9}
\]

\[
p(3) = P(WWW) = \left( \frac{2}{3} \right)^3 = \frac{8}{27}
\]

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**Example V**

You and a friend each flip a coin. If both flips are heads, your friend pays you $10; if both are tails, he pays you $5. If the coins do not match, you pay him $5. Let \( Y \) be the amount you win. What is the probability distribution for this random variable?

\[
p(0) = P(Y = 0) = 0
\]

\[
p(5) = P(TT) = \frac{1}{4}
\]

\[
p(10) = P(HH) = \frac{1}{4}
\]

\[
p(-5) = P(HT, TH) = \frac{1}{2}
\]