V. Independence

Formula and intuition

- **Definition**: Events A and B are <u>independent</u> if the following formula holds:

$$\boxed{P(A) = P(A|B)}$$

- **Intuition**: Knowing that B is true would not change the probability that A is true.

Common misinterpretations

- **Myth**: "One event affects the other in some physical way."
- **Truth**: They can be physically related or unrelated, and still be independent or not.
- **Myth**: "A and B are mutually disjoint."
- **Truth**: If they're disjoint, then they are <u>dependent</u>, since knowing that one is true tells you that the other is false.

Combining independent events

- Recall the formula for conditional probability:

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

- If A and B are independent, then $P(A|B) = P(A)$, so we get $P(A) = \dfrac{P(A \cap B)}{P(B)}$ and

$$\boxed{P(A \cap B) = P(A)P(B)}.$$

Example I

You roll two dice, one red and one blue. Consider the following events:

- A = "The red die is a three."
- B = "The blue die is a four."
- C = "The total is seven."

Are A and B independent? A and C? B and C?

- $P(A) = \frac{1}{6} = P(A|B)$. $\boxed{\text{Yes}}$
- $P(A) = \frac{1}{6} = P(A|C)$. $\boxed{\text{Yes}}$
- $P(B) = \frac{1}{6} = P(B|C)$. $\boxed{\text{Yes}}$

Intuition:

- A and B are independent. TRUE. Knowing the result of the red die does not change the probabilities of the various outcomes of the blue die.
- A and C are independent. TRUE. A total of seven is equally likely no matter what the result of the red die is.
- B and C are independent. TRUE. A total of seven is equally likely no matter what the result of the blue die is.

Example II

As before, you roll two dice, one red and one blue. Consider the following events:

- A = "The red die is a three."
- B = "The blue die is a four."
- C = "The total is seven."
- D = "The total is eight."

Are A and D independent? B and D? C and D?

- $P(A) = \frac{1}{6} \neq \frac{1}{5} = P(A|D)$. $\boxed{\text{No}}$
- $P(B) = \frac{1}{6} \neq \frac{1}{5} = P(B|D)$. $\boxed{\text{No}}$
- $P(C) = \frac{1}{6} \neq 0 = P(C|D)$. $\boxed{\text{No}}$

Intuition:

- A and D are independent. FALSE. If the red die is three, then we know that it is not one, so it is slightly more likely that the total will be eight. (If the red die were one, then the total could not be eight.)
- B and D are independent. FALSE. If the blue die is four, then we know that it is not one, so it is slightly more likely that the total will be eight.
- C and D are independent. FALSE. If the total is seven, then we know for sure that it is not eight.

Example III

In a class of ten students, a teacher randomly chooses students to present Problem 1 and Problem 2 at the blackboard. Define the following events:

A = "The teacher picks Sally for #1."
B = "The teacher picks Tom for #2."

Are A and B independent?

Depends on whether I have said I will choose different volunteers. If we have replacement, they're $\boxed{\text{independent}}$. If I must pick different volunteers, they're $\boxed{\text{not independent}}$, because A being true makes B more likely.
Solution with replacement: $P(A) = \frac{1}{10}, P(B) = \frac{1}{10}, P(A \cap B) = \frac{1}{100} = P(A)P(B)$. They're independent.
Solution without replacement: $P(A) = \frac{1}{10}, P(B) = \frac{1}{10}, P(A \cap B) = \frac{1}{10 \cdot 9} \neq P(A)P(B)$. They're not independent!

Example IV

We roll two dice. Define the following events:

$$A = \text{"The total is even."}$$
$$B = \text{"The total is } \geq 11.\text{"}$$

Are A and B independent?

Computations: $P(A) = \frac{1}{2} \neq \frac{1}{3} = P(A|B)$.
Intution: If we have at least 11, then it must be 11 or 12. 11 is more likely than 12, so the probability of A went down.
So they are $\boxed{\text{not independent}}$.

Example V

A teacher picks a student at random from a class of 14 girls and 12 boys. Define the following events:

$$\begin{aligned} A &= \text{“The teacher picks a girl.”} \\ B &= \text{“The teacher picks a boy.”} \end{aligned}$$

Are A and B independent?

Computations: $P(A) = \dfrac{14}{26} \neq 0 = P(A|B)$.
Intuition: If we picks a girl, then we didn't pick a boy. So we got new information.
So they are $\boxed{\text{not independent}}$.