XXXII. Moment-Generating Functions

Premise

- We have several random variables, Y_1, Y_2, etc.
- We want to study functions of them: $U(Y_1, \ldots, Y_n)$.
- Before, we calculated the mean of U and the variance, but that's not enough to determine the whole distribution of U.

Goal

- We want to find the full distribution function $F_U(u) := P(U \le u)$.
- Then we can find the density function $f_U(u) = F'_U(u)$.
- We can calculate probabilities:

$$P(a \le U \le b) = \int_a^b f_U(u)\,du = F_U(b) - F_U(a)$$

Three methods

1. Distribution functions. (Two lectures ago, using geometric methods from Calculus III.)
2. Transformations. (Previous lecture, using methods from Calculus I.)

3. Moment-generating functions. (This lecture.)

Review of Moment-Generating Functions

- **Recall**: The moment-generating function for a random variable Y is

$$m_Y(t) := E\left(e^{tY}\right).$$

- The MGF is a function of t (not y).

See previous lecture on MGFs.

MGFs for the Discrete Distributions

Distribution	**MGF**
Binomial	$\left[pe^t + (1-p)\right]^n$
Geometric	$\dfrac{pe^t}{1-(1-p)e^t}$
Negative binomial	$\left[\dfrac{pe^t}{1-(1-p)e^t}\right]^r$
Hypergeometric	No closed-form MGF.
Poisson	$e^{\lambda\left(e^t-1\right)}$

All are functions of t.
In the first three, we could substitute $q := 1 - p$.

MGFs for the Continuous Distributions

Distribution	**MGF**
Uniform	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$e^{\mu t + \frac{t^2\sigma^2}{2}}$
Gamma	$(1 - \beta t)^{-\alpha}$
Exponential	$(1 - \beta t)^{-1}$
Chi-square	$(1 - 2t)^{-\frac{\nu}{2}}$
Beta	No closed-form MGF.

Note that exponential is just gamma with $\alpha := 1$, and chi-square is gamma with $\alpha := \frac{\nu}{2}$ and $\beta := 2$.

Useful Formulas with MGFs

- Let $Z := aY + b$. Then

$$\boxed{m_Z(t) = e^{bt} m_Y(at)}.$$

- Suppose Y_1 and Y_2 are independent variables and $Z := Y_1 + Y_2$. Then

$$\boxed{m_Z(t) = m_{Y_1}(t) m_{Y_2}(t)}$$

How to use MGFs

- Given a function $U(Y_1, \ldots, Y_n)$, find its MGF $m_U(t)$.
- Use the useful formulas on the previous slide.
- Then compare it against your charts to see if you recognize it as a known distribution.

Example I

Let Y be a standard normal variable. Find the density function of $U := Y^2$.

$$
\begin{aligned}
m_{Y^2}(t) &:= E\left[e^{tY^2}\right] \\
&:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ty^2} e^{-\frac{y^2}{2}}\, dy
\end{aligned}
$$

Combine exponents:

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}(1-2t)}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\left(\frac{1}{1-2t}\right)}}\, dy
\end{aligned}
$$

To simplify this, recall that $\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\, dy = \int (\text{normal density}) = 1$ for any temporary σ. Take temporary $\sigma := \frac{1}{\sqrt{1-2t}}$:

$$
\begin{aligned}
&= \sigma \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\, dy \\
&= \sigma \\
&= \boxed{(1-2t)^{-\frac{1}{2}}}
\end{aligned}
$$

From your chart of mgfs, this is $\boxed{\text{chi-square with } \nu = 1}$, so we have the density:

$$
\textbf{Gamma}: f(y) := \frac{y^{\alpha-1} e^{-\frac{y}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}, 0 \le y < \infty
$$

χ^2 is gamma with $\alpha := \frac{\nu}{2}$, $\beta := 2$.

$$
\begin{aligned}
f_U(u) &= \frac{u^{-\frac{1}{2}} e^{-\frac{u}{2}}}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)}, u > 0 \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\
&= \boxed{\frac{u^{-\frac{1}{2}} e^{-\frac{u}{2}}}{\sqrt{2\pi}}, u > 0}
\end{aligned}
$$

That's one reason why chi-square is important.

Example II

Let Y_1, Y_2 be independent standard normal variables. Find the density function of $U := Y_1^2 + Y_2^2$.

$$\begin{aligned} m_U(t) &= m_{Y_1^2+Y_2^2}(t) \\ &= m_{Y_1^2}(t) m_{Y_2^2}(t) \\ &= (1-2t)^{-\frac{1}{2}}(1-2t)^{-\frac{1}{2}} \quad \text{from above} \\ &= \boxed{(1-2t)^{-1}} \end{aligned}$$

Looking at the chart, this is $\boxed{\text{chi-square with } \nu = 2}$.

$$\textbf{Gamma} : f(y) := \frac{y^{\alpha-1} e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)}, 0 \le y < \infty$$

χ^2 is gamma with $\alpha := \dfrac{\nu}{2}$, $\beta := 2$.

So $\boxed{f_U(u) = \dfrac{e^{-\frac{u}{2}}}{2}, u > 0}$. (It's also exponential with $\beta = 2$.)

$\boxed{\text{In general, if } Z_1, \ldots, Z_n \sim N(0,1), \text{ then } \sum_{i=1}^{n} Z_i^2 \sim \chi^2(n).}$

That's one reason why chi-square is important.

Example III

Let $Y_1, \ldots, Y_r$ be independent binomial variables representing $n_1, \ldots, n_r$ flips of a coin that comes up heads with probability p. Find the probability function of $U := Y_1 + \cdots + Y_r$.

Note that $0 \le u \le n_1 + \cdots + n_r$.

$$\begin{aligned} m_{Y_i}(t) &= \left[pe^t + (1-p)\right]^{n_i} \\ m_U(t) &:= m_{Y_1+\cdots+Y_r}(t) \\ &= m_{Y_1}(t) \cdots m_{Y_r}(t) \\ &= \left[pe^t + (1-p)\right]^{n_1} \cdots \left[pe^t + (1-p)\right]^{n_r} \\ &= \left[pe^t + (1-p)\right]^{n_1+\cdots+n_r} \end{aligned}$$

This is binomial with probability p and $n := n_1 + \cdots + n_r$, so

$$\boxed{p(u) = \binom{n_1 + \cdots + n_r}{u} p^u q^{n_1+\cdots+n_r-y}, 0 \le u \le n_1 + \cdots + n_r}.$$

Example IV

Let Y_1 and Y_2 be independent Poisson variables with means λ_1 and λ_2. Find the probability function of $U := Y_1 + Y_2$.

$$\begin{aligned} m_{Y_i}(t) &= e^{\lambda_i\left(e^t-1\right)} \\ m_U(t) &:= m_{Y_1+Y_2}(t) \\ &= m_{Y_1}(t)m_{Y_2}(t) \\ &= e^{\lambda_1\left(e^t-1\right)}e^{\lambda_2\left(e^t-1\right)} \\ &= e^{(\lambda_1+\lambda_2)\left(e^t-1\right)} \end{aligned}$$

This is Poisson with mean $\lambda_1 + \lambda_2$. (If you expect to see λ_1 cars at an intersection and λ_2 trucks, you expect to see $\lambda_1 + \lambda_2$ vehicles total.)

$$\begin{aligned} p(y) &= \frac{\lambda^y e^{-\lambda}}{y!} \\ p(u) &= \boxed{\frac{(\lambda_1+\lambda_2)^u e^{-\lambda_1-\lambda_2}}{u!}, 0 \le u < \infty} \end{aligned}$$

Example V

Let $Y_1, \ldots, Y_n$ be independent normal variables, each with mean μ and variance σ^2. Find the distribution of $\overline{Y} := \frac{1}{n}(Y_1 + \cdots + Y_n)$.

Let $Y := Y_1 + \cdots + Y_n$.

$$\begin{aligned} m_Y(t) &= m_{Y_1}(t) \cdots m_{Y_n}(t) \\ &= \left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)^n \\ &= e^{\mu n t + \frac{\sigma^2 t^2 n}{2}} \\ m_{aY+b}(t) &= e^{bt} m_Y(at) \\ m_{\overline{Y}} &= m_Y\left(\frac{t}{n}\right) = e^{\mu t + \frac{\sigma^2 t^2}{2n}} \end{aligned}$$

This is a $\boxed{\text{normal distribution with mean } \mu\text{, variance } \dfrac{\sigma^2}{n}}$.

Example VI

Let Y_1 and Y_2 be independent exponential variables, each with mean 3. Find the density function of $U := Y_1 + Y_2$.

$$
\begin{aligned}
m_{Y_i}(t) &= (1-3t)^{-1} \\
m_U(t) &:= m_{Y_1+Y_2}(t) \\
&= m_{Y_1}(t)m_{Y_2}(t) \\
&= (1-3t)^{-1}(1-3t)^{-1} = (1-3t)^{-2}
\end{aligned}
$$

This is gamma with $\alpha = 2, \beta = 3$:

$$
\begin{aligned}
f_U(u) &= \frac{u^{\alpha-1}e^{-\frac{u}{\beta}}}{\beta^\alpha \Gamma(\alpha)} \\
&= \frac{ue^{-\frac{u}{3}}}{3^2\Gamma(2)} \\
&= \boxed{\frac{1}{9}ue^{-\frac{u}{3}}, 0 \le u < \infty}
\end{aligned}
$$