XXXI. Transformations

Premise

- We have several random variables, Y_1, Y_2, etc.
- We want to study functions of them: $U(Y_1, \ldots, Y_n)$.
- Before, we calculated the mean of U and the variance, but that's not enough to determine the whole distribution of U.

Goal

- We want to find the full distribution function $F_U(u) := P(U \le u)$.
- Then we can find the density function $f_U(u) = F_U'(u)$.
- We can calculate probabilities:

$$P(a \le U \le b) = \int_a^b f_U(u)\,du = F_U(b) - F_U(a)$$

Three methods

1. Distribution functions. (Last lecture, using geometric methods from Calculus III.)
2. Transformations. (This lecture, using methods from Calculus I.)

3. Moment-generating functions. (Next lecture.)

Requirements for Transformation Method

- The transformation method only works for single-variable situations, that is, $U = h(Y)$.
- h must be a strictly monotonic function, which means strictly increasing or strictly decreasing.
- **Example**: All linear functions $U := aY + b$ qualify (unless $a = 0$).
- If h is monotonic, then it is invertible: We can say that $Y = h^{-1}(U)$.

Not $h(Y_1, Y_2)$.
[Graph monotonic.]
Be careful: This is an inverse function, not an exponent.

Formula for Transformations

- First solve $U = h(Y)$ for $Y = h^{-1}(U)$.
- Then use the density function $f_Y(y)$ for Y, to get the density function for U:

$$\boxed{f_U(u) = f_Y\left(h^{-1}(u)\right)\left|\frac{d}{du}h^{-1}(u)\right|}$$

Example I

Let Y have density function $f_Y(y) := \frac{3}{2}y^2$, $-1 \leq y \leq 1$. Determine whether the function $U := 3 - 2Y$ is monotonic, and if so, find its inverse.

(Graph.) First note that $1 \leq u \leq 5$.
Linear implies monotonic. (Graph.)

$$\begin{aligned} U &= h(Y) = 3 - 2Y \\ 2Y &= 3 - U \\ Y &= \boxed{\frac{3-U}{2} = h^{-1}(U)} \end{aligned}$$

Example II

As in Example I, let Y have density function $f_Y(y) := \frac{3}{2}y^2$, $-1 \leq y \leq 1$. Find the density function for $U := 3 - 2Y$.

(Graph.) Note that $1 \le u \le 5$.

$$\boxed{f_U(u) = f_Y\left(h^{-1}(u)\right)\left|\frac{d}{du}h^{-1}(u)\right|}$$

$$\begin{aligned}
U &= 3 - 2Y =: h(Y) \\
Y &= h^{-1}(U) = \frac{3-U}{2} \\
f_U(u) &= f_Y\left(h^{-1}(u)\right)\left|\frac{d}{du}h^{-1}(u)\right| \\
&= \frac{3}{2}\left(\frac{3-u}{2}\right)^2\left|-\frac{1}{2}\right| \\
&= \boxed{\frac{3}{4}\left(\frac{3-u}{2}\right)^2, 1 \le u \le 5}
\end{aligned}$$

This agrees with (but was quicker than) our earlier solution using distribution functions (Example I of previous lecture), sans integration.

Example III

Let Y have density function $f_Y(y) := \frac{3}{2}y^2$, $-1 \le y \le 1$. Determine whether the function $U := Y^2$ is monotonic, and if so, find its inverse.

(Graph.) U is not monotonic on $[-1, 1]$, so we cannot find its inverse. (It would be monotonic on $[0, 1]$.)

Example IV

Major earthquakes in California occur once every two decades on average, according to an exponential distribution. The magnitude of an earthquake is $1 + Y^2$, where Y is the time since the last earthquake. Find the density function for the magnitude of the next earthquake.

$\beta = 2 \implies f_Y(y) = \frac{1}{2}e^{-\frac{y}{2}}, 0 \le y < \infty$. $U = Y^2 + 1 =: h(Y)$. Note that this is <u>increasing</u> on $y \in [0, \infty)$. (We couldn't use this method if Y were normally distributed!) $Y = h^{-1}(U) = \sqrt{U-1}$. (If this were negative, take absolute value of it.)

$$\boxed{f_U(u) = f_Y\left(h^{-1}(u)\right)\left|\frac{d}{du}h^{-1}(u)\right|}$$

$$\begin{aligned}
f_U(u) &= f_Y\left(h^{-1}(u)\right)\left|\frac{d}{du}h^{-1}(u)\right|, 1 \le u < \infty \\
&= \frac{1}{2}e^{-\frac{\sqrt{u-1}}{2}}\frac{1}{2\sqrt{u-1}} \\
&= \boxed{\frac{1}{4\sqrt{u-1}}e^{-\frac{\sqrt{u-1}}{2}}}
\end{aligned}$$

Save this for use in Example V.

Example V

Use the density function found in Example IV to find the expected magnitude of the next earthquake. Check your answer using the properties of the exponential distribution.

$$
\begin{aligned}
f_U(u) &= \frac{1}{4\sqrt{u-1}} e^{-\frac{\sqrt{u-1}}{2}} \\
E(U) &:= \int_1^{\infty} \frac{u}{4\sqrt{u-1}} e^{-\frac{\sqrt{u-1}}{2}}\, du \qquad \text{Let } s := \frac{\sqrt{u-1}}{2},\ ds = \frac{1}{4\sqrt{u-1}}\, du. \\
&= \int_0^{\infty} \left(4s^2+1\right) e^{-s}\, ds \qquad \text{Use parts.} \\
&= \boxed{9}
\end{aligned}
$$

As a check, $E(U) = E\left(Y^2\right) + 1 = \sigma^2 + E(Y)^2 + 1 = \beta^2 + \beta^2 + 1 = \boxed{9}$. [So prepare yourself for a huge earthquake!]

Example VI

Let Y have a beta distribution with $\alpha = \beta = 2$, and let $U := Y^2 + 2Y + 1$. Find the density function $f_U(u)$, including the range of possible values for u.

$$
\begin{aligned}
f_Y(y) &:= \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} \\
&= \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} y(1-y) \\
&= 6\left(y-y^2\right)
\end{aligned}
$$

$$
\begin{aligned}
U &= h(Y) = (Y+1)^2 \\
h^{-1}(u) &= \sqrt{U}-1 \\
f_U(u) &= f_Y\left(h^{-1}(u)\right)\left|\frac{d}{du}h^{-1}(u)\right| \\
&= f_Y\left(\sqrt{u}-1\right)\frac{1}{2\sqrt{u}} \\
&= 6\left[\sqrt{u}-1-\left(\sqrt{u}-1\right)^2\right]\frac{1}{2\sqrt{u}} \\
&= 3\left(\sqrt{u}-1-u+2\sqrt{u}-1\right)\frac{1}{\sqrt{u}} \\
&= \boxed{3\left(3-\sqrt{u}-\frac{2}{\sqrt{u}}\right), 1 \le u \le 4}
\end{aligned}
$$

Check: $E(U) := \int_{u=1}^{u=4} 3u\left(3-\sqrt{u}-\frac{2}{\sqrt{u}}\right)\,du = \boxed{\frac{23}{10}}$