XXX. Distribution Functions

Premise

- We have several random variables, Y_1, Y_2, etc.
- We want to study functions of them: $U(Y_1, \ldots, Y_n)$.
- Before, we calculated the mean of U and the variance, but that's not enough to determine the whole distribution of U.

Goal

- We want to find the full distribution function $F_U(u) := P(U \leq u)$.
- Then we can find the density function $f_U(u) = F'_U(u)$.
- We can calculate probabilities:

$$P(a \leq U \leq b) = \int_a^b f_U(u)\,du = F_U(b) - F_U(a)$$

Three methods

1. Distribution functions. (This lecture, using geometric methods from Calculus III.)
2. Transformations. (Next lecture, using methods from Calculus I.)

3. Moment-generating functions. (Following lecture.)

Distribution Functions

- To find $F_U(u) := P(U \leq u)$, figure out what values of Y_1 (and/or Y_2, if it's part of it) lead to U being $\leq u$.
- Then find the probability that Y_1 (and Y_2) will be in that range (region).
- For single variables, this usually involves solving an integral.
- For two variables, this usually involves some geometry and/or solving a double integral.

Example I

Let Y have density function $f_Y(y) := \frac{3}{2}y^2$, $-1 \leq y \leq 1$. Find the density function for $U := 3 - 2Y$.

(Graph.) First note that $1 \le u \le 5$.
We want to find $P(U \le u)$. For what values of Y is $U \le u$?

$$\begin{aligned} U \le u \quad &\text{iff} \quad 3 - 2Y \le u \\ &\text{iff} \quad -2Y \le u - 3 \\ &\text{iff} \quad Y \ge \frac{3-u}{2} \end{aligned}$$

$$\begin{aligned} P(U \le u) &= P\left(Y \ge \frac{3-u}{2}\right) = \int_{y=\frac{3-u}{2}}^{y=1} \frac{3}{2} y^2 \, dy \\ &= \frac{1}{2}\left[1 - \left(\frac{3-u}{2}\right)^3\right] \end{aligned}$$

We just found $F_U(u)$, the distribution function. Note that $F_U(1) = 0$, $F_U(5) = 1$.

$$\begin{aligned} f_U(u) &= F_U'(u) \\ &= -\frac{3}{2}\left(\frac{3-u}{2}\right)^2\left(-\frac{1}{2}\right) \\ &= \boxed{\frac{3}{4}\left(\frac{3-u}{2}\right)^2, 1 \le u \le 5} \end{aligned}$$

This is the density function.

Example II

As in Example I, let Y have density function $f_Y(y) := \frac{3}{2}y^2$, $-1 \le y \le 1$. Find the density function for $U := Y^2$.

(Graph.) Note that $0 \leq u \leq 1$.
For what values of Y is $U \leq u$?

$$
\begin{aligned}
Y^2 \leq u \quad &\text{iff} \quad -\sqrt{u} \leq Y \leq \sqrt{u} \\
P(U \leq u) &= P\left(-\sqrt{u} \leq Y \leq \sqrt{u}\right) = \int_{y=-\sqrt{u}}^{y=\sqrt{u}} \frac{3}{2} y^2 \, dy \\
&= u^{\frac{3}{2}}
\end{aligned}
$$

This is $F_U(u)$.

$$f_U(u) = F_U'(u) = \boxed{\frac{3}{2}\sqrt{u}, 0 \leq u \leq 1}$$

This is the density function.

Example III

Let $f(y_1, y_2) := e^{-y_2}, 0 \leq y_1 \leq y_2 < \infty$. Find the cumulative distribution and density functions for $U := Y_1 + Y_2$.

$$
\begin{aligned}
F_U(u) &= P(U \le u) \\
&= P\left(Y_1 + Y_2 \le u\right) \qquad \text{[Graph.]} \\
&= \int_{y_1=0}^{y_1=\frac{u}{2}} \int_{y_2=y_1}^{y_2=u-y_1} e^{-y_2}\, dy_2\, dy_1 \\
&= \int_{y_1=0}^{y_1=\frac{u}{2}} -e^{-y_2}\Big|_{y_2=y_1}^{y_2=u-y_1}\, dy_1 \\
&= \int_{y_1=0}^{y_1=\frac{u}{2}} \left(e^{-y_1} - e^{y_1-u}\right) dy_1 \\
&= \int_{y_1=0}^{y_1=\frac{u}{2}} \left(e^{-y_1} - e^{-u}e^{y_1}\right) dy_1 \\
&= \left(-e^{-y_1} - e^{-u}e^{y_1}\right)\Big|_{y_1=0}^{y_1=\frac{u}{2}} \\
&= -e^{-\frac{u}{2}} - e^{-u}e^{\frac{u}{2}} + 1 + e^{-u} \\
&= \boxed{1 + e^{-u} - 2e^{-\frac{u}{2}}} \\
f_U(u) &= F_U'(u) \\
&= \boxed{e^{-\frac{u}{2}} - e^{-u}}
\end{aligned}
$$

Example IV

Let Y_1 and Y_2 have joint density function $f\left(y_1, y_2\right) :\equiv 1$ on the triangle bounded by $(0,0)$, $(2,0)$, and $(2,1)$. Find the density function for $U := Y_1 - Y_2$.

(Graph.) Note that $0 \le u \le 2$.
For what values of Y_1, Y_2 is $U \le u$?

$$\begin{aligned} Y_1 - Y_2 &\le u \\ y_2 &\ge y_1 - u \end{aligned}$$

[Graph this as a diagonal line with slope 1 and shade the area above it.] Note that $0 \le u \le 2$.
We need to integrate f on this area. But since $f \equiv 1$ (that won't happen in every problem), that just means finding the area.
$\underline{0 \le u \le 1}$:
Height: Where do the lines intersect? Plug $Y_1 = 2Y_2$ into $Y_1 - Y_2 = u$, so $Y_2 = \underline{u}$.
Base: $Y_2 = 0$, so $Y_1 = \underline{u}$.
Area: $\dfrac{u^2}{2}$
$\underline{1 \le u \le 2}$:
Area $= 1 -$ (area of little triangle cut off) $= 1 - \dfrac{1}{2}(2-u)^2$.

$$\begin{aligned} F_U(u) &= \begin{cases} \dfrac{u^2}{2}, & 0 \le u \le 1 \\ 1 - \dfrac{(2-u)^2}{2}, & 1 \le u \le 2 \end{cases} \\ f_U(u) &= F'_U(u) \\ &= \boxed{\begin{cases} u, & 0 \le u \le 1 \\ 2-u, & 1 \le u \le 2 \end{cases}} \end{aligned}$$

Example V

Let Y_1 and Y_2 be independent exponential variables with mean 1, and let U be their average.

Find the density function for U.

$$\begin{aligned}
f_1(y_1) &= e^{-y_1} \\
f_2(y_2) &= e^{-y_2} \\
F_U(u) &= P(U \leq u) = P\left(\frac{Y_1+Y_2}{2} \leq u\right) \qquad \text{[Graph } y_1+y_2=2u.\text{]} \\
&= \int_{y_1=0}^{y_1=2u} \int_{y_2=0}^{y_2=2u-y_1} e^{-y_1}e^{-y_2}\,dy_2\,dy_1 \\
&= -\int_{y_1=0}^{y_1=2u} e^{-y_1}e^{-y_2}\Big|_{y_2=0}^{y_2=2u-y_1}\,dy_1 \\
&= \int_{y_1=0}^{y_1=2u} e^{-y_1}\left(1-e^{y_1-2u}\right)\,dy_1 \\
&= \int_{y_1=0}^{y_1=2u} \left(e^{-y_1}-e^{-2u}\right)\,dy_1 \\
&= \left(-e^{-y_1}-y_1e^{-2u}\right)\Big|_{y_1=0}^{y_1=2u} \\
&= -e^{-2u}-2ue^{-2u}+1 \\
F_U(u) &= 1-2ue^{-2u}-e^{-2u} \\
f_U(u) &= 4ue^{-2u}-2e^{-2u}+2e^{-2u} \\
f_U(u) &= \boxed{4ue^{-2u}}
\end{aligned}$$