III. Choices: Combinations and permutations

Choices: With or without replacement?

- When you are choosing several things, can you choose the same thing more than once? If so, you are choosing with replacement.
- **Example**: You buy bagels at a bakery. You can buy more than one blueberry bagel if you want. This is choosing with replacement.
- **Example**: You select five players for a basketball team from a pool of 20 candidates. You can not pick the same person more than once. This is choosing without replacement.

Choices: Ordered or unordered?

- When you are choosing several things, does the order in which you select them matter? If so, you are making an ordered choice.
- **Example**: You select five players for a basketball team for a pickup game in the park. All five will run on the court together, without organizing positions. This is an unordered choice.
- **Example**: You select five players for a basketball team for a formal game with fixed positions: center, power forward, small

forward, shooting guard, and point guard. This is an ordered choice.

Selecting Kobe Bryant to play center is different from selecting him to play point guard.

Combinations

- We use combinations to count the number of ways to choose a group of r unordered objects from n possibilities without replacement:

$$\boxed{\binom{n}{r} = C_r^n = \frac{n!}{r!(n-r)!}}$$

- **Example**: Select five players for a basketball team from a pool of 20 candidates, for an informal pickup game. There are $\binom{20}{5} = \frac{20!}{5! \cdot 15!}$ ways to do this.

Pickup game in the park.

Permutations

- We use permutations to count the number of ways to choose a group of r ordered objects from n possibilities without replacement:

$$\boxed{P_r^n = \frac{n!}{(n-r)!}}$$

- **Example**: Select five players for a basketball team from a pool of 20 candidates, for a formal game with fixed positions: center, power forward, etc. There are $P_5^{20} = \frac{20!}{15!}$ ways to do this.

center, power forward, small forward, point guard, and point guard.

Key formulas

- Number of ways to pick r things from n possibilities

	With replacement	Without
Ordered	n^r	P_r^n
Unordered	$C_r^{n+r-1} = \binom{n+r-1}{r}$	$C_r^n = \binom{n}{r}$

Fill in combination formulas:

	With replacement	Without
Ordered	n^r	P_r^n
Unordered	$C_r^{n+r-1} = \binom{n+r-1}{r}$	$C_r^n = \binom{n}{r}$

Example I

We have five different candy bars to distribute to 20 children. We do not want to give any child more than one candy bar. Is this selection with replacement or without? Is it unordered or ordered? How many ways can we distribute the candy?

We are choosing $r = 5$ children from $n = 20$ possibilities. No child gets more than one, so this is $\boxed{\text{without replacement}}$. The candy bars are different, so this is $\boxed{\text{ordered}}$. There are

$$P_r^n = P_5^{20} = \frac{20!}{15!} = \boxed{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}$$

ways.

Example II

We have five identical candy bars to distribute to 20 children. We do not want to give any child more than one candy bar. Is this selection with replacement or without? Is it unordered or ordered? How many ways can we distribute the candy?

We are choosing $r = 5$ children from $n = 20$ possibilities. No child gets more than one, so this is $\boxed{\text{without replacement}}$. The candy bars are identical, so this is $\boxed{\text{unordered}}$. There are

$$\binom{n}{r} = C_r^n = C_5^{20} = \frac{20!}{5! \cdot 15!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \boxed{19 \cdot 2 \cdot 17 \cdot 16}$$

ways.

Example III

We have five identical candy bars to distribute to 20 children. We are willing to give some children more than one candy bar. Is this selection with replacement or without? Is it unordered or ordered? How many ways can we distribute the candy?

We are choosing $r = 5$ children from $n = 20$ possibilities. Children can get more than one, so this is $\boxed{\text{with replacement}}$. The candy bars are identical, so this is $\boxed{\text{unordered}}$. There are

$$\binom{n+r-1}{r} = C_r^{n+r-1} = C_5^{24} = \frac{24!}{5! \cdot 19!} = \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \boxed{6 \cdot 23 \cdot 22 \cdot 21}$$

ways.

Example IV

We have five different candy bars to distribute to 20 children. We are willing to give some children more than one candy bar. Is this selection with replacement or without? Is it unordered or ordered? How many ways can we distribute the candy?

We are choosing $r = 5$ children from $n = 20$ possibilities. Children can get more than one, so this is $\boxed{\text{with replacement}}$. The candy bars are different, so this is $\boxed{\text{ordered}}$. There are

$$\boxed{n^r = 20^5}$$

ways. We can also easily see this by just making 5 choices in a row with 20 possibilities each.

Example V

We want to buy 10 pizzas from a restaurant that carries three styles (cheese, pepperoni, and vegan). Is this selection with replacement or without? Is it unordered or ordered? How many different kinds of orders can we make?

We are choosing $r = 10$ children from $n = 3$ possibilities. We can get more than one of each style, so this is $\boxed{\text{with replacement}}$. We'll come back with ten pizzas jumbled together, so this is $\boxed{\text{unordered}}$.
There are

$$\binom{n+r-1}{r} = C_r^{n+r-1} = C_{10}^{12} = \frac{12!}{10! \cdot 2!} = \frac{12 \cdot 11}{2 \cdot 1} = \boxed{66}$$

ways.