XXVI. Conditional Probability and Conditional Expectation

Review of Marginal Probability

- We have an experiment with two random variables, Y_1 and Y_2.
- Recall the marginal probability functions and marginal density functions:

$$\begin{aligned} p_1(y_1) &= \sum_{y_2} p(y_1, y_2) \\ p_2(y_2) &= \sum_{y_1} p(y_1, y_2) \\ f_1(y_1) &= \int_{y_2=-\infty}^{y_2=\infty} f(y_1, y_2)\, dy_2 \\ f_2(y_2) &= \int_{y_1=-\infty}^{y_1=\infty} f(y_1, y_2)\, dy_1 \end{aligned}$$

Conditional Probability, Discrete Case

- $p(y_1|y_2)$ means $P(Y_1 = y_1|Y_2 = y_2)$.
- Conditional probability:

$$\boxed{p(y_1|y_2) = \frac{p(y_1, y_2)}{p_2(y_2)}}$$

Conditional Probability, Continuous Case

- Conditional density of Y_1 given that $Y_2 = y_2$

$$\boxed{f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)}}$$

- Interpret this as a <u>density</u> on Y_1 and calculate conditional probability:

$$\boxed{P\left(a \leq Y_1 \leq b | Y_2 = y_2\right) = \int_{y_1=a}^{y_1=b} f\left(y_1 | y_2\right)\, dy_1}$$

Conditional Expectation

- **Discrete**:

$$\boxed{E\left(Y_1 | Y_2 = y_2\right) = \sum_{y_1} y_1 p\left(y_1 | y_2\right)}$$

- **Continuous**:

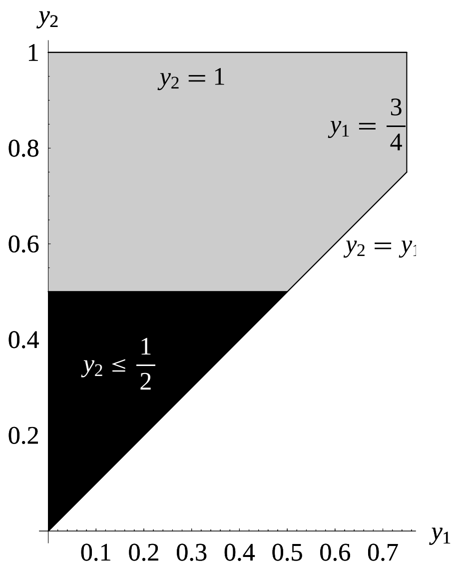
$$\boxed{E\left(Y_1 | Y_2 = y_2\right) = \int_{y_1} y_1 f\left(y_1 | y_2\right)\, dy_1}$$

Example I

Consider the joint density function

$$f\left(y_1, y_2\right) := 6\left(1 - y_2\right)$$

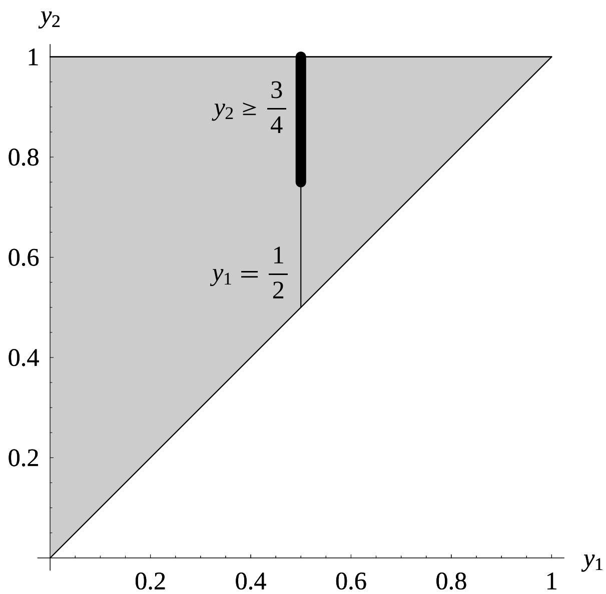
on the triangle with corners $(0, 0)$, $(0, 1)$, and $(1, 1)$. Find $P\left(Y_2 \leq \frac{1}{2} \middle| Y_1 \leq \frac{3}{4}\right)$.



$$\frac{\int_{y_1=0}^{y_1=\frac{1}{2}} \int_{y_2=y_1}^{y_2=\frac{1}{2}} 6\,(1-y_2)\ dy_2\, dy_1}{\int_{y_1=0}^{y_1=\frac{3}{4}} \int_{y_2=y_1}^{y_2=1} 6\,(1-y_2)\ dy_2\, dy_1} = \frac{\frac{1}{2}}{\frac{63}{64}} = \boxed{\frac{32}{63}}$$

Example II

As in Example I, consider the joint density function $f\,(y_1, y_2) := 6\,(1-y_2)$ on the triangle with corners $(0,0)$, $(0,1)$, and $(1,1)$. Find $P\left(Y_2 \geq \frac{3}{4} \middle| Y_1 = \frac{1}{2}\right)$.



$$
\begin{aligned}
f_1(y_1) &= \int_{y_2=y_1}^{y_2=1} 6\left(1-y_2\right)\,dy_2 = 3y_1^2 - 6y_1 + 3 \\
f_1\left(\frac{1}{2}\right) &= \frac{3}{4} \\
f\left(y_2 \middle| Y_1 = \frac{1}{2}\right) &= \frac{f\left(y_1, y_2\right)}{f_1\left(y_1\right)} = 8\left(1-y_2\right) \\
P\left(Y_2 \geq \frac{3}{4} \middle| Y_1 = \frac{1}{2}\right) &= \int_{y_2=\frac{3}{4}}^{y_2=1} 8\left(1-y_2\right)\,dy_2 = \boxed{\frac{1}{4}}
\end{aligned}
$$

Example III

Let $f\left(y_1, y_2\right) := 4y_1y_2, 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1$.
Find $P\left(Y_1 \geq \frac{3}{4} \middle| Y_2 = \frac{1}{2}\right)$.

(Graph.)

$$\begin{aligned} f_2(y_2) &:= \int_{y_1=0}^{y_1=1} 4y_1y_2\, dy_1 \\ &= 2y_2 \\ P\left(Y_1 \geq \frac{3}{4} \middle| Y_2 = \frac{1}{2}\right) &= \int_{y_1=\frac{3}{4}}^{y_1=1} f(y_1|y_2)\, dy_1 \\ &= \int_{y_1=\frac{3}{4}}^{y_1=1} \frac{f(y_1, y_2)}{f_2(y_2)}\, dy_1 \\ &= \int_{y_1=\frac{3}{4}}^{y_1=1} \frac{4y_1y_2}{2y_2}\, dy_1 \\ &= y_1^2\Big|_{y_1=\frac{3}{4}}^{y_1=1} \\ &= \boxed{\frac{7}{16}} \end{aligned}$$

Example IV

As in Example III, let

$$f(y_1, y_2) := 4y_1y_2, 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1.$$

Find the conditional expectation $E\left(Y_1 \middle| Y_2 = \frac{1}{2}\right)$.

$\displaystyle\int_{y_1=0}^{y_1=1} y_1 f(y_1|y_2)\,dy_1$. Last time we calculated $f(y_1|y_2) = 2y_1$. (In general, this might have a y_2 in it. If so, plug in $y_2 = \frac{1}{2}$.)

$$\begin{aligned}\int_{y_1=0}^{y_1=1} y_1 f(y_1|y_2)\,dy_1 &= \int_{y_1=0}^{y_1=1} y_1 2y_1\,dy_1 \\ &= \left.\frac{2}{3}y_1^3\right|_{y_1=0}^{y_1=1} \\ &= \boxed{\frac{2}{3}}\end{aligned}$$

Example V

Consider the joint density function

$$f(y_1, y_2) := \begin{cases} e^{-y_2}, & 0 \le y_1 \le y_2 < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the conditional expectation $E(Y_2|Y_1 = 5)$.

$$
\begin{aligned}
f_1(y_1) &= \int_{y_2=y_1}^{y_2=\infty} e^{-y_2}\,dy_2 \\
&= \left(-e^{-y_2}\right)\Big|_{y_2=y_1}^{y_2=\infty} \\
&= e^{-y_1} \\
f_1(5) &= e^{-5} \\
f(y_2|Y_1=5) &= \frac{f(y_1,y_2)}{e^{-5}} \\
&= e^{5-y_2} \\
E(Y_2|Y_1=5) &= \int_{y_2=5}^{y_2=\infty} y_2 e^{5-y_2}\,dy_2 \\
&= e^5 \int_{y_2=5}^{y_2=\infty} y_2 e^{-y_2}\,dy_2 \quad \text{Use parts.} \\
&= e^5 \left(-y_2 e^{-y_2} - e^{-y_2}\right)\Big|_{y_2=5}^{y_2=\infty} \\
&= e^5 \left(0 - 0 + 5e^{-5} + e^{-5}\right) \\
&= \boxed{6}
\end{aligned}
$$