XXVI. Conditional Probability and Conditional Expectation

Review of Marginal Probability

- We have an experiment with two random variables, \( Y_1 \) and \( Y_2 \).
- Recall the marginal probability functions and marginal density functions:

\[
\begin{align*}
    p_1(y_1) &= \sum_{y_2} p(y_1, y_2) \\
    p_2(y_2) &= \sum_{y_1} p(y_1, y_2) \\
    f_1(y_1) &= \int_{y_2=-\infty}^{y_2=\infty} f(y_1, y_2) \, dy_2 \\
    f_2(y_2) &= \int_{y_1=-\infty}^{y_1=\infty} f(y_1, y_2) \, dy_1
\end{align*}
\]

Conditional Probability, Discrete Case

- \( p(y_1 | y_2) \) means \( P(Y_1 = y_1 | Y_2 = y_2) \).
- Conditional probability:

\[
p(y_1 | y_2) = \frac{p(y_1, y_2)}{p_2(y_2)}
\]

Conditional Probability, Continuous Case

- Conditional density of \( Y_1 \) given that \( Y_2 = y_2 \)

\[
f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)}
\]
• Interpret this as a density on $Y_1$ and calculate conditional probability:

$$ P(a \leq Y_1 \leq b | Y_2 = y_2) = \int_{y_1=a}^{y_1=b} f(y_1 | y_2) \, dy_1 $$

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**Conditional Expectation**

• Discrete:

$$ E(Y_1 | Y_2 = y_2) = \sum_{y_1} y_1 p(y_1 | y_2) $$

• Continuous:

$$ E(Y_1 | Y_2 = y_2) = \int_{y_1} y_1 f(y_1 | y_2) \, dy_1 $$

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**Example I**

Consider the joint density function

$$ f(y_1, y_2) := 6 (1 - y_2) $$

on the triangle with corners $(0,0)$, $(0,1)$, and $(1,1)$. Find $P \left( Y_2 \leq \frac{1}{2} \middle| Y_1 \leq \frac{3}{4} \right)$. 

Example II

As in Example I, consider the joint density function \( f(y_1, y_2) := 6(1 - y_2) \) on the triangle with corners \((0, 0)\), \((0, 1)\), and \((1, 1)\). Find \[ P\left( Y_2 \geq \frac{3}{4} \mid Y_1 = \frac{1}{2} \right). \]
Example III

Let \( f(y_1, y_2) := 4y_1y_2, 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1. \)

Find \( P \left( Y_1 \geq \frac{3}{4} \mid Y_2 = \frac{1}{2} \right). \)
(Graph.)

\[
\begin{align*}
    f_2(y_2) &:= \int_{y_1=0}^{y_1=1} 4y_1y_2 \, dy_1 \\
    &= 2y_2 \\
    P \left( Y_1 \geq \frac{3}{4} \, \bigg| \, Y_2 = \frac{1}{2} \right) &= \int_{y_1=\frac{3}{4}}^{y_1=1} f(y_1 \, | \, y_2) \, dy_1 \\
    &= \int_{y_1=\frac{3}{4}}^{y_1=1} \frac{f(y_1, y_2)}{f_2(y_2)} \, dy_1 \\
    &= \int_{y_1=\frac{3}{4}}^{y_1=1} \frac{4y_1y_2}{2y_2} \, dy_1 \\
    &= \left[ y_1^2 \right]_{y_1=\frac{3}{4}}^{y_1=1} \\
    &= \frac{7}{16}
\end{align*}
\]

Example IV

As in Example III, let

\[ f(y_1, y_2) := 4y_1y_2, \quad 0 \leq y_1 \leq 1, \quad 0 \leq y_2 \leq 1. \]

Find the conditional expectation

\[ E \left( Y_1 \, \bigg| \, Y_2 = \frac{1}{2} \right). \]
\[ \int_{y_2 = 0}^{y_1 = 1} y_1 f(y_1|y_2) \, dy_1. \]

Last time we calculated
\[ f(y_1|y_2) = 2y_1. \]
(In general, this might have a \( y_2 \) in it. If so, plug in \( y_2 = \frac{1}{2} \).

\[
\int_{y_1 = 0}^{y_1 = 1} y_1 f(y_1|y_2) \, dy_1 = \int_{y_1 = 0}^{y_1 = 1} y_1 2y_1 \, dy_1 \\
= \frac{2}{3} y_1^3 \bigg|_{y_1 = 0}^{y_1 = 1} \\
= \frac{2}{3}
\]

Example V

Consider the joint density function
\[
f(y_1, y_2) := \begin{cases} 
  e^{-y_2}, & 0 \leq y_1 \leq y_2 < \infty, \\
  0, & \text{elsewhere.}
\end{cases}
\]

Find the conditional expectation \( E(Y_2|Y_1 = 5) \).
\[ f_1(y_1) = \int_{y_2=y_1}^{y_2=\infty} e^{-y_2} dy_2 \]
\[ = \left. -e^{-y_2} \right|_{y_2=\infty}^{y_2=y_1} \]
\[ = e^{-y_1} \]
\[ f_1(5) = e^{-5} \]
\[ f(y_2|Y_1 = 5) = \frac{f(y_1, y_2)}{e^{-5}} \]
\[ = e^{5-y_2} \]
\[ E(Y_2|Y_1 = 5) = \int_{y_2=5}^{y_2=\infty} y_2 e^{5-y_2} dy_2 \]
\[ = e^5 \int_{y_2=5}^{y_2=\infty} y_2 e^{-y_2} dy_2 \quad \text{Use parts.} \]
\[ = e^5 \left. \left( -y_2 e^{-y_2} - e^{-y_2} \right) \right|_{y_2=\infty}^{y_2=5} \]
\[ = e^5 \left( 0 - 0 + 5e^{-5} + e^{-5} \right) \]
\[ = 6 \]