XXII. Beta Distribution

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Beta Function

- Fixed parameters: $\alpha > 0, \beta > 0$
- We define the beta function
  \[ B(\alpha, \beta) := \int_{0}^{1} y^{\alpha-1}(1-y)^{\beta-1} \, dy = \text{some constant} \]
- Relationship between the gamma and beta functions:
  \[ B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \]
- This makes it easy to compute for whole numbers, since $\Gamma(n) = (n-1)!$.

(Plug in values for $\alpha$ and $\beta$, and it spits out a number.)

Don’t mix up the beta function and the beta distribution!

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Beta Distribution

- We want to define a distribution on $[0,1]$ using the function
  \[ y^{\alpha-1}(1 - y)^{\beta-1} \]
• Recall that a probability density function must satisfy \( \int_0^1 f(y) \, dy = 1 \). To make this work, we divide by \( B(\alpha, \beta) \):

• Density function for the beta distribution:

\[
f(y) := \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)}, 0 \leq y \leq 1
\]

Don’t mix up the beta function and the beta distribution!

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**Key Properties of the Beta Distribution**

• Mean:

\[
\mu = E(Y) = \frac{\alpha}{\alpha + \beta}
\]

• Variance:

\[
\sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}
\]

• Standard deviation:

\[
\sigma = \sqrt{V(Y)} = \sqrt{\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}}
\]

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**Example I**

Calculate \( B(3, 4) \).
\[ B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \]
\[ B(3, 4) = \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{2! \cdot 3!}{6!} = \frac{2 \cdot 6}{720} = \frac{1}{60} \]

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**Example II**

On the same set of axes, graph the density functions for the beta distribution for the following combinations of \((\alpha, \beta)\):

\( \left( \frac{1}{2}, 2 \right) \) \( (1, 4) \) \( (10, 2) \) \( (1.1, 2) \)
\[ f(y) := \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)} \]

Look at \( y = 0 \) and recall the behavior of the gamma density function near \( y = 0 \).

- If \( \alpha > 1 \), it goes to 0.
- If \( \alpha = 1 \), it goes to 1. (Actually it scales to some other finite nonzero number.)
- If \( \alpha < 1 \), it goes to \( \infty \).

So \( \alpha \) controls the behavior at \( y = 0 \).

Similarly, \( \beta \) controls what it does at \( y = 1 \).

- Red: \( \alpha = \frac{1}{2} \) (goes to \( \infty \)), \( \beta = 2 \) (goes to 0).
- Green: \( \alpha = 1 \) (finite limit), \( \beta = 4 \) (pulls down to 0). [I took a bigger \( \beta \) to make it not so linear.]
- Black: \( \alpha = 10 \) (pulls it down strongly), \( \beta = 2 \) (must get the area somewhere).
- Blue: \( \alpha = 1.1 \) (like \( \alpha = 1 \), but with a dropoff right at 0), \( \beta = 2 \) (line \( 1 - y \)).
Example III

Show that the uniform distribution is a special case of the beta distribution.

Take $\alpha := 1, \beta := 1$.

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

$$B(1, 1) = \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} = 1$$

$$f(y) := \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)} \equiv 1 \quad 0 \leq y \leq 1$$

Example IV

Show that the distribution with triangular density function $f(y) := 2y$, $0 \leq y \leq 1$ is a special case of the beta distribution.

Take $\alpha := 2, \beta := 1$.

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

$$B(1, 1) = \frac{\Gamma(2)\Gamma(1)}{\Gamma(3)} = \frac{1}{2}$$

$$f(y) := \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)} = 2y, 0 \leq y \leq 1$$

Example V
The length of your morning commute (in hours) is a random variable $Y$ that has a beta distribution with $\alpha = \beta = 2$.

A. Find the chance that your commute tomorrow will take longer than 30 minutes.

B. Your rage level is $R := Y^2 + 2Y + 1$. Find the expected value of $R$.

\[
f(y) := \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)}
\]
\[= \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)}y(1-y)
\]
\[= 6 (y - y^2)
\]

Example V

\[f(y) = 6 (y - y^2)\]
A. 

\[ P \left( Y \geq \frac{1}{2} \right) = \int_{\frac{1}{2}}^{1} 6 \left( y - y^2 \right) \, dy \]

\[ = 6 \left( \frac{y^2}{2} - \frac{y^3}{3} \right) \bigg|_{y=\frac{1}{2}}^{y=1} \]

\[ = 6 \left( \frac{1}{2} - \frac{1}{3} \right) \left( \frac{1}{8} + \frac{1}{24} \right) \]

\[ = 3 - 2 - \frac{3}{4} + \frac{1}{4} = \frac{1}{2} \]

B. 

\[ E(Y) = \frac{\alpha}{\alpha + \beta} = \frac{1}{2} \]

\[ E(Y^2) = V(Y) + E(Y)^2 \]

\[ = \frac{\alpha \beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} + \left( \frac{\alpha}{\alpha + \beta} \right)^2 \]

\[ = \frac{4}{16 \cdot 5} + \frac{1}{4} = \frac{3}{10} \]

\[ E(R) = \frac{3}{10} + 2 \cdot \frac{1}{2} + 1 = \frac{23}{10} \]