XX. Normal (Gaussian) Distribution

Normal (Gaussian) Distribution

- The normal distribution (also known as the Gaussian distribution) is the most common continuous distribution. It is the famous "bell curve".
- Fixed parameters:

$$\begin{aligned} \mu &:= \text{mean} \\ \sigma &:= \text{standard deviation} \end{aligned}$$

Formula for the Normal Distribution

- Density function:

$$\boxed{f(y) := \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, -\infty < y < \infty}$$

- $P(a \leq Y \leq b) = \displaystyle\int_a^b f(y)\,dy$, but this can't be integrated in general.

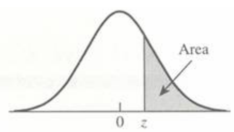
Standard Normal Distribution

- The standard normal distribution has mean $\mu = 0$ and standard deviation $\sigma = 1$.
- We use $Z \sim N(0, 1)$ to indicate that Z has the standard normal distribution.

- We can find probabilities for a standard normal variable Z using charts, calculators, computer algebra systems, or online applications.

Standard Normal Distribution

Normal curve areas
Standard normal probability in right-hand tail (for negative values of z areas are found by symmetry)



	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681

Nonstandard Normal Distribution

- When you have a nonstandard normal variable $Y \sim N\left(\mu, \sigma^2\right)$, then you can form the associated standard normal:

$$\boxed{Z := \frac{Y - \mu}{\sigma} \sim N(0,1)}$$

- Convert your range for Y into a range for Z:

$$P(a \le Y \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$

- Then you can look up probabilities for Z.

Example I

What is the chance that a standard normal variable will land between 1 and 2?

	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
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0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
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0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183

$$\begin{aligned} P(1 \le Z \le 2) &= P(Z \ge 1) - P(Z \ge 2) \\ &\approx 0.1587 - 0.0228 \\ &= \boxed{0.1359} \end{aligned}$$

Example II

If a set of data is normally distributed, then what proportion of the data lies within two standard deviations of the mean?

	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
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1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
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1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183

$1 - 2(0.0228) \approx \boxed{95.44\%}$ of the data lies within two standard deviations of the mean.)

Example III

Scores on an exam are normally distributed with a mean of 76 and variance of 64.

A. What proportion of scores are between 72 and 96?

B. The minimum passing score is 60. Find the proportion of students that will pass.

A. $Z := \dfrac{Y-76}{8}$ is a standard normal variable.

$$\begin{aligned} P(72 \leq Y \leq 96) &= P(-4 \leq Y - 76 \leq 20) \\ &= P\left(-0.5 \leq \frac{Y-76}{8} \leq 2.5\right) \\ &\approx (0.5 - 0.3085) + (0.5 - .0062) \quad \text{from the standard normal table} \\ &= \boxed{68.53\%} \end{aligned}$$

B.

$$\begin{aligned} P(Y > 60) &= P(Y - 76 \leq -16) \\ &= P\left(\frac{Y-76}{8} \leq -2.0\right) \\ &\approx 1 - 0.0228 \quad \text{from the standard normal table} \\ &= \boxed{97.72\%} \end{aligned}$$

Example IV

Daytime high temperatures in Long Beach are normally distributed with a mean of 75 and a standard deviation of 9. What percentage of days have high temperatures over 88?

$\mu = 75, \sigma = 9$. We want $P(Y \geq 88)$. $Z := \dfrac{Y - 75}{9}$ is standard normal:

$$\begin{aligned} z(88) &= \frac{88-75}{9} \\ &= \frac{13}{9} \\ &\approx 1.44 \end{aligned}$$

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z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
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0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
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1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183

$$\text{Chart:} \quad P(Z \geq 1.44) \quad \approx \quad 0.0749$$

So the probability is $\approx 0.0749 = \boxed{7.49\%}$.

Example V

Scores on an exam are normally distributed with a mean of 70 and variance of 16.

A. What percentage of the scores are between 68 and 78?

B. What is the minimum score to be in the top 10% of students?

Let $Z := \frac{Y-70}{4}$, a standard normal variable.

A. $68 < Y < 78$ iff $-\frac{1}{2} < Z < 2$, which occurs with probability $\approx 1 - 0.3085 - 0.0228 \approx \boxed{67\%}$.

B. The 10% cutoff is at $Z > 1.28$, which translates into $\approx Y > \boxed{75}$.