XIX. Uniform Distribution

Uniform Distribution

- The uniform distribution is a continuous distribution on a finite range $[\theta_1, \theta_2]$ in which the density is uniformly distributed:

$$\boxed{f(y) \equiv \frac{1}{\theta_2 - \theta_1}, \theta_1 \le y \le \theta_2}$$

- Each part of the region is equally probable:

$$\boxed{P(a \le Y \le b) = \frac{b-a}{\theta_2 - \theta_1}}$$

Draw graph.

Key Properties of the Uniform Distribution

- **Mean**:

$$\boxed{\mu = E(Y) = \frac{\theta_1 + \theta_2}{2}}$$

- **Variance**:

$$\boxed{\sigma^2 = V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}}$$

- **Standard deviation**:

$$\boxed{\sigma = \sqrt{V(Y)} = \frac{\theta_2 - \theta_1}{2\sqrt{3}}}$$

The mean should be obvious. The variance is not.

Example I

Each day your newspaper arrives at a time that is uniformly distributed between 7am and noon. Find the probability that it will arrive during an odd numbered hour.

$$\frac{(8-7)+(10-9)+(12-11)}{5} = \boxed{\frac{3}{5}}$$

Example II

If you pick a real number y from a uniform distribution on [5,12], what is the probability that $y \geq 9$?

$$\frac{12-9}{12-5} = \boxed{\frac{3}{7}}$$

Example III

You arrange to meet a friend for dinner at 6pm. Both of you are chronically tardy; your arrival time is uniformly distributed between 0 and 15 minutes late, and your friend's arrival is uniformly distributed between 0 and 10 minutes late. What the probability that you will arrive before your friend?

Let

$$\begin{aligned} X &:= \text{your time} \\ Y &:= \text{your friend's time} \end{aligned}$$

Graph a rectangle $[0,15] \times [0,10]$.

$$\begin{aligned} P(X < Y) &= P(Y > X) \\ &= \frac{\text{area above } y = x}{\text{total area}} \\ &= \frac{50}{150} = \boxed{\frac{1}{3}} \end{aligned}$$

Example IV

Show that if Y is uniformly distributed on [0,1], then the variable

$$X := (\theta_2 - \theta_1)\, Y + \theta_1$$

is uniformly distributed on $[\theta_1, \theta_2]$. (This is useful for computer programmers, who can use a random number generator on [0,1] to simulate any uniform distribution on any range.)

$$\begin{aligned}
Y = 0 &\implies X = \theta_1 \\
Y = 1 &\implies X = \theta_2 \\
P(a \le X \le b) &= P\left(a \le (\theta_2 - \theta_1)\, Y + \theta_1 \le b\right) \\
&= P\left(a - \theta_1 \le (\theta_2 - \theta_1)\, Y \le b - \theta_1\right) \\
&= P\left(\frac{a - \theta_1}{\theta_2 - \theta_1} \le Y \le \frac{b - \theta_1}{\theta_2 - \theta_1}\right) \\
&= \frac{b - \theta_1}{\theta_2 - \theta_1} - \frac{a - \theta_1}{\theta_2 - \theta_1} \\
&= \frac{b - a}{\theta_2 - \theta_1}
\end{aligned}$$

So X is a uniform distribution on $[\theta_1, \theta_2]$.

Example V

An ice cream machine dispenses between 206 and 230 milliliters of ice cream, uniformly distributed. Find the expected amount of ice cream in a serving and the standard deviation.

$$\begin{aligned}
\mu = E(Y) &= \frac{\theta_1 + \theta_2}{2} = \frac{206 + 230}{2} = \boxed{218 \text{ millileters}} \\
\sigma &= \frac{\theta_2 - \theta_1}{2\sqrt{3}} = \frac{230 - 206}{2\sqrt{3}} = \frac{24}{2\sqrt{3}} = \frac{12}{\sqrt{3}} = \boxed{4\sqrt{3} \text{ millileters}}
\end{aligned}$$

(That's about 7ml.)