XVIII. Mean and Variance for Continuous Distributions

Mean

- Let Y be a continuous random variable. We can calculate its mean, also known as its expected value:

$$\mu = E(Y) := \int_{-\infty}^{\infty} y f(y)\, dy$$

- As in the discrete case, expectation is linear:
  1. $E(c) = c$
  2. $E(Y_1 + Y_2) = E(Y_1) + E(Y_2)$
  3. $E(cY) = cE(Y)$

Variance

- By definition, the variance is

$$\begin{aligned} \sigma^2 = V(Y) &:= E\left[(Y-\mu)^2\right] \\ &:= \int_{-\infty}^{\infty} (y-\mu)^2 f(y)\, dy. \end{aligned}$$

- However, as in the discrete case, it is usually easier to calculate via the mean:

$$\begin{aligned} \sigma^2 &= E\left(Y^2\right) - E(Y)^2 \\ &= \left(\int_{-\infty}^{\infty} y^2 f(y)\, dy\right) - \mu^2. \end{aligned}$$

Standard Deviation

- As in the discrete case, we compute the standard deviation from the variance:

$$\sigma := \sqrt{V(Y)}$$

Example I

As in Example III of the previous video, let Y have density function

$$f(y) := \begin{cases} \frac{1}{3}, & 0 \le y \le 1, \\ \\ \frac{2}{3}, & 1 < y \le 2. \end{cases}$$

Find $E(Y)$.

$$\begin{aligned} E(Y) &:= \int_0^2 y f(y)\, dy \\ &:= \frac{1}{3}\left(\int_0^1 y\, dy + \int_1^2 2y\, dy\right) \\ &= \frac{1}{3}\left(\frac{y^2}{2}\Big|_{y=0}^{y=1} + y^2\Big|_{y=1}^{y=2}\right) \\ &= \frac{1}{3}\left(\frac{1}{2} + 4 - 1\right) = \boxed{\frac{7}{6}} \end{aligned}$$

Example II

Let $\theta_1 < \theta_2$ be constants and consider the uniform density function

$$f(y) = \frac{1}{\theta_2 - \theta_1}, \theta_1 \leq y \leq \theta_2.$$

Find $E(Y)$.

(We predict $\mu = \boxed{\frac{\theta_1 + \theta_2}{2}}$.)

$$\begin{aligned} \mu = E(Y) &:= \int_{-\infty}^{\infty} y f(y)\, dy \\ &= \int_{\theta_1}^{\theta_2} y \frac{1}{\theta_2 - \theta_1}\, dy \\ &= \frac{1}{\theta_2 - \theta_1} \left. \frac{y^2}{2} \right|_{y=\theta_1}^{y=\theta_2} \\ &= \frac{\theta_2^2 - \theta_1^2}{2(\theta_2 - \theta_1)} \\ \mu &= \boxed{\frac{\theta_1 + \theta_2}{2}} \end{aligned}$$

Example III

Let $\theta_1 < \theta_2$ be constants and consider the uniform density function

$$f(y) = \frac{1}{\theta_2 - \theta_1}, \theta_1 \leq y \leq \theta_2.$$

Find $V(Y)$.

$$
\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} y^2 f(y)\, dy - \mu^2 \\
&= \int_{\theta_1}^{\theta_2} y^2 \frac{1}{\theta_2 - \theta_1}\, dy - \left(\frac{\theta_1 + \theta_2}{2}\right)^2 \\
&= \frac{1}{\theta_2 - \theta_1} \left. \frac{y^3}{3} \right|_{y=\theta_1}^{y=\theta_2} - \left(\frac{\theta_1 + \theta_2}{2}\right)^2 \\
&= \frac{\theta_2^3 - \theta_1^3}{3\,(\theta_2 - \theta_1)} - \left(\frac{\theta_1 + \theta_2}{2}\right)^2 \\
&= \frac{\theta_2^2 + \theta_2\theta_1 + \theta_1^2}{3} - \frac{\theta_1^2 + 2\theta_1\theta_2 + \theta_2^2}{4} \\
&= \frac{\theta_2^2 - 2\theta_2\theta_1 + \theta_1^2}{12} \\
\sigma^2 &= \boxed{\frac{(\theta_2 - \theta_1)^2}{12}}
\end{aligned}
$$

Example IV

Let Y have density function

$$
f(y) := \begin{cases} \frac{1}{2}(2 - y), & 0 \le y \le 2, \\ 0, & \text{elsewhere.} \end{cases}
$$

Find $E(Y)$.

$$
\begin{aligned}
\mu = E(Y) &:= \int_{-\infty}^{\infty} y f(y)\, dy \\
&= \int_0^2 \frac{1}{2} y(2-y)\, dy \\
&= \int_0^2 \left(y - \frac{1}{2} y^2 \right) dy \\
&= \left(\frac{y^2}{2} - \frac{y^3}{6} \right) \Bigg|_{y=0}^{y=2} \\
&= 2 - \frac{8}{6} = \frac{6}{3} - \frac{4}{3} = \boxed{\frac{2}{3}}
\end{aligned}
$$

Example V

Let Y have density function

$$
f(y) := \begin{cases} \frac{1}{2}(2-y), & 0 \le y \le 2, \\ 0, & \text{elsewhere.} \end{cases}
$$

Find $V(Y)$.

$$
\begin{aligned}
E\left(Y^2\right) &:= \int_{-\infty}^{\infty} y^2 f(y)\,dy \\
&= \int_0^2 \frac{1}{2} y^2 (2-y)\,dy \\
&= \int_0^2 \left(y^2 - \frac{1}{2}y^3\right)\,dy \\
&= \left(\frac{y^3}{3} - \frac{y^4}{8}\right)\Bigg|_{y=0}^{y=2} \\
&= \frac{8}{3} - \frac{16}{8} = \frac{8}{3} - 2 = \frac{2}{3} \\
\sigma^2 &= E\left(Y^2\right) - E(Y)^2 \quad E(Y) = \frac{2}{3} \text{ from Example IV.} \\
&= \frac{2}{3} - \frac{4}{9} = \frac{6}{9} - \frac{4}{9} = \boxed{\frac{2}{9}}
\end{aligned}
$$