XVI. Poisson Distribution

Poisson Distribution

- The Poisson distribution describes events that occur randomly and independently, such as calls coming in to a tech support center.
- Y := number of calls in an hour

Real examples: How many calls come in to tech support in one hour? Cars pass a checkpoint on a country road? Major earthquakes strike in the next 20 years? Soldiers in the Prussian army die from horse kicks?

Formula for the Poisson Distribution

- Fixed parameter:

$$\begin{aligned}\lambda &:= \text{average number of calls per hour} \\ &\quad\ \text{(doesn't have to be an integer)}\end{aligned}$$

- Probability distribution:

$$\boxed{p(y) = \frac{\lambda^y}{y!}e^{-\lambda}, 0 \leq y < \infty}$$

Key Properties of the Poisson Distribution

- **Mean**:

$$\boxed{\mu = E(Y) = \lambda}$$

- **Variance**:

$$\boxed{\sigma^2 = V(Y) = \lambda}$$

- **Standard deviation**:

$$\boxed{\sigma = \sqrt{V(Y)} = \sqrt{\lambda}}$$

The mean should be obvious. The variance is not.

Example I

California averages 6 major forest fires per year. What is the chance that there will be exactly 4 fires this year? What is the chance that there will be at least 4 fires?

$$
\begin{aligned}
p(y) &= \frac{\lambda^y}{y!}e^{-\lambda} \\
p(4) &= \frac{6^4}{4!}e^{-6} = \boxed{\frac{54}{e^6} \approx 0.13385 \approx 13.4\%} \\
P(Y \geq 4) &= 1 - p(0) - p(1) - p(2) - p(3) \\
&= 1 - e^{-6}\left(1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6}\right) \\
&= 1 - e^{-6}\left(1 + 6 + \frac{6^2}{2} + \frac{6^3}{6}\right) \\
&= 1 - e^{-6}(7 + 18 + 36) \\
&= \boxed{1 - \frac{61}{e^6}} \\
&\approx \boxed{84.9\%}
\end{aligned}
$$

Example II

A call center receives two calls per minute on average.

A. Use Markov's inequality to estimate the chance that fewer than 5 calls will come in in the next minute.

B. Find the exact chance that fewer than 5 calls will come in in the next minute.

Example II

$\lambda = 2$

A. Markov, $P(Y < 5)$.

B. Exact, $P(Y < 5)$.

A. Markov says

$$\begin{aligned} P(Y \geq a) &\leq \frac{E(Y)}{a}. \\ P(Y \geq 5) &\leq \frac{2}{5} = 40\% \\ P(Y < 5) &\geq 60\% \end{aligned}$$

So $\boxed{\text{there is at least a 60\% chance that fewer than 5 calls will come in}}$.

B.

$$\begin{aligned} p(y) &= \frac{\lambda^y}{y!}e^{-\lambda} \\ P(Y \leq 4) &= P(0) + P(1) + P(2) + P(3) + P(4) \\ &= e^{-2}\left(1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right) \\ &= e^{-2}\left(3 + 2 + \frac{4}{3} + \frac{2}{3}\right) \\ &= \boxed{\frac{7}{e^2}} \approx \boxed{94.7\%} \end{aligned}$$

Example III

Use the definition of expected value to confirm that the mean of the Poisson distribution is λ.

$$\begin{aligned}
E(Y) &:= \sum_{y=0}^{\infty} y p(y) = \sum y \frac{\lambda^y}{y!} e^{-\lambda} = ? \\
\text{Let} \quad f(\lambda) &:= \sum \frac{\lambda^y}{y!} = e^\lambda. \\
\text{Take } \frac{d}{d\lambda}: \quad f'(\lambda) &= \sum y \frac{\lambda^{y-1}}{y!} = e^\lambda \\
\text{Multiply by } \lambda: \quad \lambda f'(\lambda) &= \boxed{\sum y \frac{\lambda^y}{y!} = \lambda e^\lambda} \\
\text{Multiply by } e^{-\lambda}: \quad & \sum y \frac{\lambda^y}{y!} e^{-\lambda} = \lambda e^\lambda e^{-\lambda} \\
& \boxed{E(Y) = \lambda}
\end{aligned}$$

Example IV

Find $E\left(Y^2\right)$ for the Poisson distribution.

$$\begin{aligned}
\sigma^2 &= E\left(Y^2\right) - E(Y)^2 = \lambda \\
E\left(Y^2\right) - E(Y)^2 &= \lambda \\
E\left(Y^2\right) - \lambda^2 &= \lambda \\
E\left(Y^2\right) &= \boxed{\lambda^2 + \lambda}
\end{aligned}$$

Example V

California averages two major earthquakes per decade. Let Y represent the number of major earthquakes in California in the next decade.

The cost of damage (in millions of dollars) is determined to be $C = 2Y^2 + 5Y + 10$.

A. Find the expected cost.

B. Find the probability that damages will cost more than \$40 million.

Example V

$\lambda = 2, C = 2Y^2 + 5Y + 10$.

A. Find the expected cost.

B. Find $P(C \geq 40)$.

A.

$$\begin{aligned} E(C) &= 2E\left(Y^2\right) + 5E(Y) + 10 \\ &= 2\left(\lambda^2 + \lambda\right) + 5\lambda + 10 \quad \text{From Example IV} \\ &= 2\lambda^2 + 7\lambda + 10 = 8 + 14 + 10 = \boxed{\$32 \text{ million}} \end{aligned}$$

B.

$$\begin{aligned} C &> 40 \\ 2Y^2 + 5Y + 10 &> 40 \\ 2Y^2 + 5Y &> 30 \\ \text{Trial and error:} \quad Y &\geq 3 \\ P(Y \geq 3) &= 1 - p(0) - p(1) - p(2) \\ &= 1 - e^{-2}\left(1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6}\right) \\ &= 1 - e^{-2}\left(1 + 2 + \frac{2^2}{2}\right) \\ &= \boxed{1 - \frac{5}{e^2}} \approx \boxed{32.3\%} \end{aligned}$$