XI. Tchebysheff's Inequality

Tchebysheff's Inequality

- Tchebysheff's inequality is a quick way of estimating probabilities based on the mean μ and standard deviation σ of a random variable.
- Suppose Y is a random variable, and k is constant.

$$\boxed{\textbf{Tchebysheff}: P(|Y-\mu| > k\sigma) \leq \frac{1}{k^2}}$$

- **Intuition**: It is unlikely that the variable will be far (many standard deviations away) from its mean.

Tchebysheff's Inequality in Reverse

- Suppose Y is a random variable, and k is constant.

$$\boxed{\textbf{Tchebysheff}: P(|Y-\mu| > k\sigma) \leq \frac{1}{k^2}}$$

- We can reverse it:

$$\boxed{P(|Y-\mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}}$$

- **Intuition**: It is likely that the variable will be close to (within a few standard deviations of) its mean.

Example I

Surveys show that students on a particular campus carry an average of \$20 in cash, with a standard deviation of \$10. If you meet a student at random, estimate the chance that she is carrying more than \$100. Also estimate the chance that she is carrying less than \$80.

Tchebysheff says

$$\begin{aligned} P(|Y-\mu| > k\sigma) &\leq \frac{1}{k^2} \\ P(|Y-20| > 8\cdot 10) &\boxed{\leq} \frac{1}{8^2} = \boxed{\frac{1}{64}} \\ P(|Y-\mu| \leq k\sigma) &\geq 1-\frac{1}{k^2} \\ P(|Y-20| < 6\cdot 10) &\boxed{\geq} 1-\frac{1}{6^2} = \boxed{\frac{35}{36}} \end{aligned}$$

Example II

Students on a college campus have completed an average of 50 units, with a standard deviation of 15 units. Estimate the chance that a randomly selected student has completed more than 95 units.

Tchebysheff says

$$P(|Y-\mu|>k\sigma) \leq \frac{1}{k^2}$$

$$P(|Y-50|>45) = P(|Y-50|>3\sigma) \boxed{\leq \frac{1}{3^2}=\frac{1}{9}}$$

Example III

Scores on a national exam are symmetrically distributed around a mean of 76 with a variance of 64. The minimum passing score is 60. Use Tchebysheff's inequality to estimate the proportion of students that will pass.

$$P(|Y-\mu|>k\sigma) \leq \frac{1}{k^2}$$

$$P(|Y-76|>8k) \leq \frac{1}{k^2}$$

$$k=2: \quad P(|Y-76|>16) \leq \frac{1}{4}$$

$$P(Y<60)+(Y>92) \leq \frac{1}{4}$$

$$\text{By symmetry} \quad (Y<60) \leq \frac{1}{8}$$

$$P(Y>60) > \frac{7}{8}$$

So $\boxed{\text{at least } 87.5\% \text{ of the students will pass}}$.

Example IV

Seismic data indicate that California suffers a major earthquake on average once every 10 years, with a standard deviation of 10 years. What can we say about the probability that there will be an earthquake in the next 30 years?

$\mu = \sigma = 10$. Tchebysheff tells us that

$$P(|Y - \mu| \geq 2\sigma) \leq \frac{1}{4},$$

so the probability that there will be one is $\boxed{\geq \frac{3}{4}}$.

Example V

Housing prices in Smalltown are symmetrically distributed with a mean of \$50,000 and a standard deviation of \$20,000. Use Tchebysheff's inequality to estimate the proportion of houses that cost less than \$90,000.

If I ever do this again, I should change the numbers so that it's not identical to Example III.

By Tchebysheff, at least 75% are between \$10K and \$90K, so at least 37.5% are between \$50K and \$90K. So $\boxed{\geq 87.5\%}$ are below \$90K.