VIII. Second order equations: repeated roots and reduction of order

---

Lesson Overview

- To solve the (linear, second-order, homogeneous, constant coefficient) differential equation

$$ay'' + by' + cy = 0$$

first solve the characteristic equation:

$$ar^2 + br + c = 0$$

- If the characteristic equation has a double root r, then the general solution to the differential equation is

$$\boxed{y_{\text{gen}} = c_1 e^{rt} + c_2 t e^{rt}}.$$

- As before, to find c_1 and c_2, use initial conditions, usually given as $y(0)$ and $y'(0)$. You'll get two equations in two unknowns.

- The second solution was found by a method called reduction of order. If you have one solution $y_1(t)$ to a (second-order linear homogeneous) differential equation

$$y''(t) + p_1(t)y'(t) + p_2(t)y(t) = 0$$

then you can find a second solution $y_2(t) = v(t)y_1(t)$, where $v(t) = \int w(t)\,dt$ and $w(t)$ is a solution to the first-order equation

$$y_1 w' + (2y_1' + p_1 y_1)\, w = 0.$$

Example I

Find the general solution to the differential equation:

$$y'' - 4y' + 4y = 0$$

$r = 2, 2 \Longrightarrow$

General Solution: $\boxed{y_{\text{gen}} = c_1 e^{2t} + c_2 t e^{2t}}$.

Example II

Solve the initial value problem:

$$y'' - 4y' + 4y = 0, y(0) = 3, y'(0) = 8$$

General Solution from before:

$y_{\text{gen}} = \; c_1 e^{2t} + c_2 t e^{2t} \Longrightarrow y(0) = c_1 = 3$

$y' = 2c_1 e^{2t} - 2c_1 t e^{2t} + c_2 e^{2t} \Longrightarrow y'(0) = 2c_1 + c_2 = 8$

$2(3) + c_2 = 8 \Longrightarrow c_2 = 2$

$y = \boxed{3e^{2t} + 2te^{2t}}$

Example III

Consider the differential equation:

$$ty'' - 2(t+1)y' + 4y = 0$$

A. Check that $y_1 = e^{2t}$ is a solution to the equation.

B. Use reduction of order to find a second (independent) solution.

A.

$$\begin{aligned} y &= e^{2t} \\ y' &= 2e^{2t} \\ y'' &= 4e^{2t} \\ ty'' - 2(t+1)y' + 4y &= 4te^{2t} - 2(t+1)2e^{2t} + 4e^{2t} \\ &= (4t - 4t - 4 + 4)e^{2t} = 0\checkmark \end{aligned}$$

Example III

$$ty'' - 2(t+1)y' + 4y = 0, y_1 = e^{2t}$$

B. $y_2(t) = v(t)y_1(t)$, where $v(t) = \int w(t)\,dt$ and $w(t)$ is a solution to the first-order equation

$$y_1 w' + (2y_1' + p_1 y_1)\, w = 0, p_1 = \frac{-2(t+1)}{t}.$$

$$
\begin{aligned}
e^{2t}w' + \left(4e^{2t} + \frac{-2(t+1)}{t}e^{2t}\right)w &= 0 \\
w' + \left(4 + \frac{-2(t+1)}{t}\right)w &= 0 \\
w' + \left(4 - 2 - \frac{2}{t}\right)w &= 0 \\
w' + \left(2 - \frac{2}{t}\right)w &= 0 \\
w' &= \left(\frac{2}{t} - 2\right)w \\
\frac{dw}{dt} &= \left(\frac{2}{t} - 2\right)w \\
\frac{dw}{w} &= \left(\frac{2}{t} - 2\right)dt \\
\ln w &= 2\ln t - 2t \\
w &= e^{2\ln t - 2t} = e^{2\ln t}e^{-2t} = t^2e^{-2t} \\
\text{Integrate by parts:} \quad v &= \int w(t)\,dt = -\frac{t^2}{2}e^{-2t} - \frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t} \\
y_2 &= e^{2t}v \\
&= \boxed{-\frac{t^2}{2} - \frac{t}{2} - \frac{1}{4}}
\end{aligned}
$$

Since this is a linear homogeneous equation, we can multiply by a constant to simplify it. The best choice is -4:

$$y_2 = \boxed{2t^2 + 2t + 1}$$

Example IV

Find the general solution to the differential equation:

$$y'' + 2y' + y = 0$$

$r = -1, -1 \Longrightarrow$

General Solution: $\boxed{y_{\text{gen}} = c_1 e^{-t} + c_2 t e^{-t}}$.

Example V

Consider the differential equation:

$$t^2 y'' - t(t+2)y' + (t+2)y = 0$$

A. Check that $y_1 = t$ is a solution to the equation.

B. Use reduction of order to find a second (independent) solution.

A.

$$\begin{aligned} y &= t \\ y' &= 1 \\ y'' &= 0 \\ t^2 y'' - t(t+2)y' + (t+2)y &= 0 - t(t+2)(1) + (t+2)(t) = 0 \checkmark \end{aligned}$$

Example V

$$t^2 y'' - t(t+2)y' + (t+2)y = 0, y_1 = t$$

B. $y_2(t) = v(t) y_1(t)$, where $v(t) = \int w(t)\, dt$ and $w(t)$ is a solution to the first-order equation

$$y_1 w' + (2y_1' + p_1 y_1)\, w = 0, p_1 = \frac{-t(t+2)}{t^2} = -\frac{t+2}{t}.$$

$$
\begin{aligned}
tw' + \left(2 - \frac{t+2}{t}t\right) w &= 0 \\
tw' + [2 - (t+2)]\, w &= 0 \\
tw' - tw &= 0 \\
w' &= w \\
\frac{dw}{dt} &= w \\
\frac{dw}{w} &= 1 \\
\ln w &= t \\
w &= e^t \\
v &= \int w(t)\, dt = e^t \\
y_2 &= \boxed{te^t}
\end{aligned}
$$