VI. Second order equations, distinct roots

Lesson Overview

- To solve the (linear, second-order, homogeneous, constant coefficient) differential equation

$$ay'' + by' + cy = 0$$

first solve the characteristic equation:

$$ar^2 + br + c = 0$$

- You'll get roots r_1 and r_2. Then the general solution is

$$\boxed{y_{\text{gen}} = c_1 e^{r_1 t} + c_2 e^{r_2 t}}.$$

- To find c_1 and c_2, use initial conditions, usually given as $y(0)$ and $y'(0)$. You'll get two equations in two unknowns.

Example I

Find the general solution to the differential equation:

$$y''(t) + 2y'(t) - 8y(t) = 0$$

$$\begin{aligned} y = e^{rt} \quad &\Longrightarrow \quad r^2 + 2r - 8 = 0 \quad \{(\underline{\text{Characteristic equation}}) \quad \} \\ r = -4, 2 \quad &\Longrightarrow \quad y = e^{-4t} \text{ or } y = e^{2t} \end{aligned}$$

General Solution: $\boxed{y_{\text{gen}} = c_1 e^{-4t} + c_2 e^{2t}}$.

Example II

Solve the initial value problem

$$y''(t) + 2y'(t) - 8y(t) = 0, y(0) = 1, y'(0) = -10$$

and describe the behavior of the solution as $t \to \infty$.

Example II

$$y''(t) + 2y'(t) - 8y(t) = 0, y(0) = 1, y'(0) = -10$$

General Solution from before:

$$\boxed{y_{\text{gen}} = c_1 e^{-4t} + c_2 e^{2t}}$$

$$\begin{aligned} y(0) = 1 &\implies c_1 + c_2 = 1 \\ y'(0) = -10 &\implies -4c_1 + 2c_2 = -10 \\ &\implies c_1 = 2, c_2 = -1 \\ &\implies \boxed{y = 2e^{-4t} - e^{2t}} \end{aligned}$$

As $t \to \infty$, $y \to -\infty$, because the e^{2t} dominates.

Example III

Find the general solution of $y'' - 4y' - 5y = 0$.

$$\boxed{y_{\text{gen}} = c_1 e^{-t} + c_2 e^{5t}}$$

Example IV

Solve the initial value problem:

$$y'' - 4y' - 5y = 0, y(0) = 4, y'(0) = 2$$

$$y_{\text{gen}} = c_1 e^{-t} + c_2 e^{5t}$$

$$c_1 + c_2 = 4, 5c_2 - c_1 = 2 \Longrightarrow c_1 = 3, c_2 = 1$$

$\boxed{y = 3e^{-t} + e^{5t}}$

Example V

Find at least one nonzero solution to the differential equation:

$$y'' + 6y' + 9y = 0$$

$$r^2 + 6r + 9 = (r+3)^2 = 0 \Longrightarrow r = -3, -3$$

So we have a solution $\boxed{y = c_1 e^{-3t}}$, but we can't (yet) find a second solution that is independent of e^{-3t}. It would be incorrect to say $\boxed{y_{\text{gen}} = c_1 e^{-3t} + c_2 e^{-3t}}$. We'll learn how to solve this in a later lecture on repeated roots.