V. Autonomous equations

Lesson Overview

- Autonomous differential equations have the form $y' = f(y)$. (No x or t appears on the right.)
- For these equations we can often understand their behavior by graphing y' versus y first. This is called phase plane analysis.
- We use the phase plane analysis to predict the shape of solutions.
- $y' = 0$ gives equilibrium solutions.
- $y' < 0$ gives us solutions that tend downward.
- $y' > 0$ gives us solutions that tend upward.
- If we perturb the equilibrium solutions, which ones will return to equilibrium?
- Solutions will return to stable equilibria. (y' is going downwards in the phase plane.)
- Solutions will tend away from unstable equilibria. (y' is going upwards in the phase plane.)
- Semistable equilibria are stable on one side and unstable on the other.

Example I

Consider the differential equation:

$$y' = y(y-1)(2-y)$$

A. Draw a graph of y' versus y.

B. Identify the equilibrium solutions.

Example II

Consider the differential equation described above:

$$y' = y(y-1)(2-y)$$

A. Sketch other solutions.

B. Label the equilibrium solutions as stable, semistable, or unstable.

C. Predict $y(\infty)$ for the following real life initial conditions:

$$y(2) = 5, y(1) = \frac{3}{2}, y(3) = 1, y(2) = -6$$

Example II

$$y' = y(y-1)(2-y)$$
$$y(2) = 5, y(1) = \frac{3}{2}, y(3) = 1, y(2) = -6$$

A. Make graphs!

B. 0 is stable. 1 is unstable. 2 is stable.

A. $y(2) = 5 \Longrightarrow y(t \to \infty) \to \boxed{2}$

B. $y(1) = \frac{3}{2} \Longrightarrow \boxed{2}$

C. $y(3) = 1 \Longrightarrow \boxed{1}$* (Unstable, so in real life it would go to 0 or 2.)

D. $y(2) = -6 \Longrightarrow \boxed{0}$

Example III

Consider the differential equation:

$$y' = (y-1)^2(y-2)(y-3)$$

A. Draw a graph of y' versus y.

B. Identify the equilibrium solutions.

Example IV

Consider the differential equation described above:

$$y' = (y-1)^2(y-2)(y-3)$$

A. Sketch other solutions.

B. Label equilibrium solutions as stable, semistable, or unstable.

C. Predict $y(\infty)$ for the following real life initial conditions:

$$y(0) = 4, y(0) = \frac{1}{2}$$

Example IV

$$y' = (y-1)^2(y-2)(y-3)$$
$$y(0) = 4, y(0) = \frac{1}{2}$$

3 is unstable. 2 is stable. 1 is semistable up. So $y(0) = 4$ goes to $\boxed{\infty}$, and $y(0) = \frac{1}{2}$ goes up to $\boxed{1}$, and in the real world, will eventually jump up to $\boxed{2}$.

Example V

Consider the differential equation:

$$y' = y^3 - 4y^2 + 5y - 2$$

A. Draw a graph of y' versus y.

B. Identify the equilibrium solutions.

Example VI

Consider the differential equation described above:

$$y' = y^3 - 4y^2 + 5y - 2$$

A. Sketch other solutions.

B. Label the equilibrium solutions as stable, semistable, or unstable.

C. Describe what ranges of values for the initial condition $y(0) = y_0$ would lead to what limiting behavior for the solution.

Example VI

Consider the differential equation described above:

$$y' = y^3 - 4y^2 + 5y - 2$$

A. The curves between $y = 1$ and $y = 2$ should be sloping down slightly.

B. $y = 1$ is a semistable equilibrium. $y = 2$ is an unstable equilibrium.

C.
- If $y_0 < 1$, then $y \to -\infty$ as $t \to \infty$.
- If $1 \leq y_0 < 2$, then $y \to 1$ as $t \to \infty$.
- If $y_0 = 2$, then $y \to 2$ as $t \to \infty$.
- If $y_0 > 2$, then $y \to \infty$ as $t \to \infty$.