XXX. Partial differential equations: Solution of the heat equation

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Lesson Overview

- We want to solve the heat equation and its boundary and initial conditions:

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= f(x), 0 \le x \le L \end{aligned}$$

Solving the heat equation

- Using separation of variables (see earlier lecture), we get the solution to the equation and boundary conditions:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}$$

- Then we try to match the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = f(x)$$

- So we need a Fourier series for $f(x)$ using only sines.

Procedure for the heat equation

1. Extend $f(x)$ on $-L \leq x \leq 0$ to be an odd function, so that its Fourier series will have only sines.

2. Find the Fourier series for $f(x)$:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

3. Plug the coefficients back into the solution above of the heat equation:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}}$$

Example I

Consider the boundary value problem below:

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ u(0,t) &= 0, t \geq 0 \\ u(L,t) &= 0, t \geq 0 \\ u(x,0) &= f(x) = 3, 0 \leq x \leq L = 3 \end{aligned}$$

Extend the initial function $f(x)$ to be an odd function.

$$f(x) = \boxed{\begin{cases} 3 & \text{if } 0 \leq x \leq 3 \\ -3 & \text{if } -3 < x < 0 \end{cases}}$$

Example II

Find a Fourier sine series for the initial function from the example above:

$$f(x) = \begin{cases} 3 & \text{if } 0 \leq x \leq 3 \\ -3 & \text{if } -3 < x < 0 \end{cases}$$

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\
&= \frac{2}{3}\int_0^3 3\sin\frac{n\pi x}{3}\,dx \\
&= 2\left(-\frac{3}{n\pi}\cos\frac{n\pi x}{3}\right)\Bigg|_{x=0}^{x=3} \\
&= \frac{6}{n\pi}(1-\cos n\pi) \\
&= \begin{cases} \frac{12}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \\
FS(x) &= \boxed{\frac{12}{\pi}\sin\frac{\pi x}{3}+\frac{12}{3\pi}\sin\frac{3\pi x}{3}+\frac{12}{5\pi}\sin\frac{5\pi x}{3}+\cdots}
\end{aligned}$$

Example III

Solve the boundary value problem from the example above:

$$\begin{aligned}
u_t &= \alpha^2 u_{xx} \\
u(0,t) &= 0, t \geq 0 \\
u(L,t) &= 0, t \geq 0 \\
u(x,0) &= f(x) = 3, 0 \leq x \leq L = 3
\end{aligned}$$

$$\begin{aligned}
b_n &= \begin{cases} \frac{12}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \\
u(x,t) &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \\
&= \frac{12}{\pi}\sin\frac{\pi x}{3}e^{-\frac{\pi^2\alpha^2}{9}t}+\frac{12}{3\pi}\sin\frac{3\pi x}{3}e^{-\frac{9\pi^2\alpha^2}{9}t}+\frac{12}{5\pi}\sin\frac{5\pi x}{3}e^{-\frac{25\pi^2\alpha^2}{9}t}+\cdots \\
&= \boxed{\sum_{n=0}^{\infty}\frac{12}{(2n+1)\pi}\sin\frac{(2n+1)\pi x}{3}e^{-\frac{(2n+1)^2\pi^2\alpha^2}{9}t}}
\end{aligned}$$

Example IV

Consider the boundary value problem below:

$$\begin{aligned} u_t &= 4u_{xx} \\ u(0,t) &= 0, t \geq 0 \\ u(4,t) &= 0, t \geq 0 \\ u(x,0) &= f(x) = x, 0 \leq x \leq L = 4 \end{aligned}$$

Extend the initial function $f(x)$ to be an odd function.

$$f(x) = \boxed{x, -4 \leq x \leq 4}$$

Example V

Find a Fourier sine series for the initial function from the example above:

$$f(x) = x, -4 \leq x \leq 4$$

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\
&= \frac{1}{2}\int_0^4 x\sin\frac{n\pi x}{4}\,dx \qquad \{\text{Integrate by parts.} \qquad\qquad \} \\
&= \frac{1}{2}\left[-\frac{4x}{n\pi}\cos\frac{n\pi x}{4}+\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]\Bigg|_{x=0}^{x=4} \\
&= \frac{1}{2}\left[-\frac{16}{n\pi}\cos n\pi+0+0-0\right] \\
&= -\frac{8}{n\pi}\cos n\pi \\
&= (-1)^{n+1}\frac{8}{n\pi} \\
FS(x) &= \boxed{\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{4}}
\end{aligned}$$

Example VI

Solve the boundary value problem from the example above:

$$\begin{aligned}
u_t &= 4u_{xx} \\
u(0,t) &= 0, t\geq 0 \\
u(4,t) &= 0, t\geq 0 \\
u(x,0) &= f(x)=x, 0\leq x\leq L=4
\end{aligned}$$

$$\begin{aligned}
b_n &= (-1)^{n+1}\frac{8}{n\pi} \\
u(x,t) &= \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \\
&= \boxed{\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{4}e^{-\frac{n^2\pi^2}{4}t}}
\end{aligned}$$