XXIX. Partial differential equations: Fourier series

Lesson Overview

- The Fourier series for a function $f(x)$ is an expansion into sines and cosines as follows:

$$FS(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

- Find the Fourier coefficients by the formulas:

$$\left. \begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \end{aligned} \right. , n = 0, 1, 2, \ldots$$

Notes on Fourier series

- For $n = 0$, the formula simplifies:

$$\boxed{a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx}$$

- The function $f(x)$ must be periodic with period $2L$, that is, $f(x + 2L) = f(x)$ for all x.

Even and odd functions

- **Definition**: A function $f(x)$ is even if $f(-x) = f(x)$ for all x. It is odd if $f(-x) = -f(x)$ for all x.
- **Examples**: $\sin x$ is odd. $\cos x$ is even. x is odd. x^2 is even, as is x^n for any even n. x^n is odd for any odd n. The zero function is both even and odd.
- A function is even iff its graph has mirror symmetry across the y-axis. A function is odd iff the graph has rotational symmetry around the origin.

Even and odd functions and Fourier series

- If f is even, then $f(x)\sin\frac{n\pi x}{L}$ is odd, so $b_n = 0$, and the Fourier Series contains only cosines:

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= 0 \end{aligned} \quad , n = 0, 1, 2, \ldots$$

- If f is odd, then $f(x)\cos\frac{n\pi x}{L}$ is odd, so $a_n = 0$, and the Fourier Series contains only sines:

$$\begin{aligned} a_n &= 0 \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned} \quad , n = 1, 2, \ldots$$

Extending functions

- If the function $f(x)$ is only defined for $0 \le x \le L$ (as in the heat equation), then we can extend $f(x)$ on $-L \le x \le 0$ to be either even or odd, whichever we like.

- If we want a cosine series, we extend $f(x)$ to be even:
$$f(-x) := f(x)$$
- If we want a sine series, we extend $f(x)$ to be odd:
$$f(-x) := -f(x)$$
- To solve the heat equation in the next lecture, we will need to use sine series, so we will extend $f(x)$ to be odd.

Example I

Find a Fourier Series for the function below:

$$f(x) = \begin{cases} 0 & \text{if } -2 < x \leq 0 \\ x & \text{if } 0 \leq x \leq 2 \\ f(x \pm 4) & \text{if } x \leq -2 \text{ or } x > 2 \end{cases}$$

Graph. Here f is periodic with period 4, and $L = 2$.

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-2}^{2} f(x)\, dx \\ &= \frac{1}{2} \int_{0}^{2} x\, dx \\ &= \frac{1}{4} x^2 \Big|_{x=0}^{x=2} \\ &= 1 \end{aligned}$$

Note that $\frac{a_0}{2}$ is always the average value of the function.

Example I

$$f(x) = \begin{cases} 0 & \text{if } -2 < x \leq 0 \\ x & \text{if } 0 \leq x \leq 2 \end{cases}$$

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx \\
&= \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx \qquad \{\text{Integrate by parts.} \qquad\qquad \} \\
&= \frac{1}{2}\left(\frac{2}{n\pi}x\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\Bigg|_{x=0}^{x=2} \\
&= \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2}\cos n\pi - 0 - \frac{4}{n^2\pi^2}\right] \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Example I

$$f(x) = \begin{cases} 0 & \text{if } -2 < x \leq 0 \\ x & \text{if } 0 \leq x \leq 2 \end{cases}$$

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx \\
&= \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx \qquad \{\text{Integrate by parts.} \qquad\qquad \} \\
&= \frac{1}{2}\left(-\frac{2}{n\pi}x\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\Bigg|_{x=0}^{x=2} \\
&= \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + 0 + 0 - 0\right] \\
&= (-1)^{n+1}\frac{2}{n\pi}
\end{aligned}$$

Conclusion: The Fourier Series for f is

$$\boxed{FS(x) = \frac{1}{2} - \frac{4}{\pi^2}\cos\frac{\pi x}{2} + \frac{2}{\pi}\sin\frac{\pi x}{2} - \frac{2}{2\pi}\sin\frac{2\pi x}{2} - \frac{4}{9\pi^2}\cos\frac{3\pi x}{2} + \frac{2}{3\pi}\sin\frac{3\pi x}{2} - \cdots}$$

Example II

Use the Fourier series derived above to find the value of the series

$$1 + \frac{1}{9} + \frac{1}{25} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

$$\boxed{FS(x) = \frac{1}{2} - \frac{4}{\pi^2}\cos\frac{\pi x}{2} + \frac{2}{\pi}\sin\frac{\pi x}{2} - \frac{2}{2\pi}\sin\frac{2\pi x}{2} - \frac{4}{9\pi^2}\cos\frac{3\pi x}{2} + \frac{2}{3\pi}\sin\frac{3\pi x}{2} - \cdots}$$

Plug in $x = 0$:

$$\begin{aligned}
f(0) &= FS(0) \\
0 &= \frac{1}{2} - \frac{4}{\pi^2} - \frac{4}{9\pi^2} - \frac{4}{25\pi^2} - \cdots \\
&= \frac{1}{2} - \frac{4}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) \\
\frac{1}{2} &= \frac{4}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) \\
\frac{\pi^2}{8} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}
\end{aligned}$$

Example III

Plug in the endpoint value $x = 2$ to the Fourier series above. What does the Fourier series converge to when the original function is discontinuous?

$$\boxed{FS(x) = \frac{1}{2} - \frac{4}{\pi^2}\cos\frac{\pi x}{2} + \frac{2}{\pi}\sin\frac{\pi x}{2} - \frac{2}{2\pi}\sin\frac{2\pi x}{2} - \frac{4}{9\pi^2}\cos\frac{3\pi x}{2} + \frac{2}{3\pi}\sin\frac{3\pi x}{2} - \cdots}$$

$$\begin{aligned}
FS(2) &= \frac{1}{2} + \frac{4}{\pi^2} + \frac{4}{9\pi^2} + \frac{4}{25\pi^2} + \cdots \\
&= \frac{1}{2} + \frac{4}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) \\
&= \frac{1}{2} + \frac{4}{\pi^2}\frac{\pi^2}{8} \qquad \{\text{by the series above} \qquad\qquad \} \\
&= \frac{1}{2} + \frac{1}{2} \\
FS(2) &= \boxed{1}
\end{aligned}$$

The series splits the difference and converges to a point exactly halfway between the left and right hand limit of the original function f!

Example IV

Extend the function below in such a way that its Fourier series will contain only cosines:

$$f(x) = 3 - x, 0 \leq x \leq 3$$

Draw graph. We want it to be even, so first we extend:

$$\boxed{f(x) := x + 3, -3 \leq x < 0}$$

$L = 3$, so we now extend it to have period $2L = 6$:

$$\boxed{f(x \pm 6) := f(x)}$$

Example V

Find a Fourier series for the function defined above that contains only cosines:

$$f(x) = \begin{cases} x+3, & -3 \le x < 0 \\ 3-x, & 0 \le x \le 3 \\ f(x \pm 6), & x \notin [-3,3] \end{cases}$$

$$\begin{aligned} a_0 &= \frac{2}{3} \int_0^3 (3-x)\, dx \\ &= 3 \end{aligned}$$

Example V

$$f(x) = \begin{cases} x+3, & -3 \le x < 0 \\ 3-x, & 0 \le x \le 3 \end{cases}$$

$$\begin{aligned} a_n &= \frac{2}{3} \int_0^3 (3-x) \cos \frac{n\pi x}{3}\, dx \\ &= \frac{2}{3} \left[9 \sin \frac{n\pi x}{3} - \frac{3}{n\pi} x \sin \frac{n\pi x}{3} - \frac{9}{n^2\pi^2} \cos \frac{n\pi x}{3} \Big|_{x=0}^{x=3} \right] \\ &= \frac{2}{3} \left(-\frac{9}{n^2\pi^2} \right) (\cos n\pi - 1) \\ &= \begin{cases} \frac{12}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \\ FS(x) &= \frac{3}{2} + \frac{12}{\pi^2} \cos \frac{\pi x}{3} + \frac{12}{9\pi^2} \cos \frac{3\pi x}{3} + \frac{12}{25\pi^2} \cos \frac{5\pi x}{3} + \cdots \\ &= \boxed{\frac{3}{2} + \frac{12}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos \frac{(2n+1)\pi x}{3}} \end{aligned}$$