XXVIII. Partial differential equations: Separation of variables

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Lesson Overview

- Separation of variables is a technique for solving some partial differential equations.
- Assume the function you're looking for, $u(x, t)$, can be written as a product of a function of x only and a function of t only:
  $$u(x, t) = X(x)T(t)$$
- Then it is easy to take derivatives:
  $$\begin{array}{ll} u_x = X'(x)T(t) & u_{xx} = X''(x)T(t) \\ u_t = X(x)T'(t) & u_{tt} = X(x)T''(t) \end{array}$$
- Plug them in to the partial differential equation.

Separation of variables

- Try to separate the variables:
  (function of x only) = (function of t only)
- If you can, then both sides must be constant:
  (function of x only) = λ = (function of t only)
- Reorganize these into two ordinary differential equations
  $$\begin{aligned} \text{(function of } x \text{ only)} &= \lambda \\ \text{(function of } t \text{ only)} &= \lambda \end{aligned}$$
  which you can solve separately for X and T.

Example I

Use separation of variables to convert the following partial differential equation into two ordinary differential equations:

$$u_{xx} + xu_t = 0$$

$$\begin{aligned}
u(x,t) &= X(x)T(t) \\
u_x &= X'(x)T(t) \\
u_{xx} &= X''(x)T(t) \\
u_t &= X(x)T'(t) \\
\text{Plug in to the PDE:} \quad X''(x)T(t) + xX(x)T'(t) &= 0 \\
-\frac{X''(x)}{xX(x)} &= \frac{T'(t)}{T(t)} = \lambda \\
\boxed{X''(x) + \lambda x X(x)} &= 0 \\
\boxed{T'(t) - \lambda T(t)} &= 0
\end{aligned}$$

Example II

Use separation of variables to convert the following partial differential equation into two ordinary differential equations:

$$u_{tt} + u_{xt} + u_x = 0$$

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ u_x &= X'(x)T(t) \\ u_{tt} &= X(x)T''(t) \\ u_{xt} &= X'(x)T'(t) \end{aligned}$$

Plug in to the PDE:

$$\begin{aligned} X(x)T''(t) + X'(x)T'(t) + X'(x)T(t) &= 0 \\ X(x)T''(t) + X'(x)\left[T'(t) + T(t)\right] &= 0 \\ X'(x)\left[T'(t) + T(t)\right] &= -X(x)T''(t) \\ -\frac{X'(x)}{X(x)} &= \frac{T''(t)}{T'(t)+T(t)} = \lambda \\ \boxed{X'(x) + \lambda X(x)} &= 0 \\ \boxed{T''(t) - \lambda T'(t) - \lambda T(t)} &= 0 \end{aligned}$$

Example III

Use separation of variables to convert the following partial differential equation into two ordinary differential equations:

$$u_{xx} + u_{tt} + tu = 0$$

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ u_{xx} &= X''(x)T(t) \\ u_{tt} &= X(x)T''(t) \end{aligned}$$

Plug in to the PDE:

$$\begin{aligned} X''(x)T(t) + X(x)T''(t) + tX(x)T(t) &= 0 \\ X''(x)T(t) + X(x)\left[T''(t) + tT(t)\right] &= 0 \\ X''(x)T(t) &= -X(x)\left[T''(t) + tT(t)\right] \\ -\frac{X''(x)}{X(x)} &= \frac{T''(t)+tT(t)}{T(t)} = \lambda \\ \boxed{X''(x) + \lambda X(x)} &= 0 \\ \boxed{T''(t) + (t-\lambda)T(t)} &= 0 \end{aligned}$$

Example IV

Use separation of variables to convert the heat equation below into two ordinary differential equations. (For later purposes, use $-\lambda$ instead of λ for the separation constant.)

$$u_t = \alpha^2 u_{xx}$$

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ u_x &= X'(x)T(t) \\ u_{xx} &= X''(x)T(t) \\ u_t &= X(x)T'(t) \\ \text{Plug in to the PDE:} \quad X(x)T'(t) &= \alpha^2 X''(x)T(t) \\ \frac{T'(t)}{\alpha^2 T(t)} &= \frac{X''(x)}{X(x)} = -\lambda \\ \boxed{\frac{T'(t)}{\alpha^2 T(t)}} &= -\lambda \\ \frac{X''(x)}{X(x)} &= -\lambda \\ \boxed{X''(x) + \lambda X(x)} &= 0 \end{aligned}$$

Example V

Solve the two ordinary differential equations below from the heat equation. Assume that $\lambda > 0$ and find solutions that satisfy the boundary conditions $u(0,t) = u(L,t) = 0, t \geq 0$.

$$\begin{aligned} \frac{T'(t)}{\alpha^2 T(t)} &= -\lambda \\ X''(x) + \lambda X(x) &= 0 \end{aligned}$$

$$
\begin{aligned}
\frac{T'(t)}{\alpha^2 T(t)} &= -\lambda \quad \{\text{Separable first order ODE.}\} \\
\frac{T'(t)}{T(t)} &= -\lambda\alpha^2 \quad \{\text{Integrate both sides.}\} \\
\ln|T(t)| &= -\lambda\alpha^2 t + C \\
T(t) &= \pm e^{-\lambda\alpha^2 t + C} \\
&= \pm e^C e^{-\lambda\alpha^2 t + C} \\
&= \boxed{ke^{-\lambda\alpha^2 t}}
\end{aligned}
$$

Example V

$$
\begin{aligned}
T(t) &= ke^{-\lambda\alpha^2 t} \\
X''(x) + \lambda X(x) &= 0
\end{aligned}
$$

$$
\begin{aligned}
X''(x) + \lambda X(x) &= 0 \quad \left\{\begin{array}{l}\text{Second order linear ODE} \\ \text{with constant coefficients.} \\ \text{Guess } X(x) = e^{rx}.\end{array}\right\} \\
r^2 + \lambda &= 0 \\
r &= \pm\sqrt{-\lambda}
\end{aligned}
$$

Suppose $\lambda < 0$. Then $-\lambda > 0$, so $r = \pm\sqrt{-\lambda}$ leads to real solutions:

$$
\begin{aligned}
X(x) &= ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x} \\
u(x,t) &= \left(ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}\right) e^{-\lambda\alpha^2 t} \quad \{\text{Absorb the } k \text{ as before.}\} \\
u(0,t) &= (a+b)e^{-\lambda\alpha^2 t} = 0 \quad \{\text{Plug in } t = 0:\} \\
u(0,0) &= \boxed{a + b = 0} \\
u(L,t) &= \left(ae^{\sqrt{-\lambda}L} + be^{-\sqrt{-\lambda}L}\right) e^{-\lambda\alpha^2 t} \\
\text{Plug in } t = 0: \quad u(L,0) &= \boxed{ae^{\sqrt{-\lambda}L} + be^{-\sqrt{-\lambda}L} = 0}
\end{aligned}
$$

Example V

$$\begin{aligned} a+b &= 0 \\ ae^{\sqrt{-\lambda}L} + be^{-\sqrt{-\lambda}L} &= 0 \end{aligned}$$

Solve the two equations for a and b:

$$\begin{aligned} a+b &= 0 \Longrightarrow b=-a \\ ae^{\sqrt{-\lambda}L} + be^{-\sqrt{-\lambda}L} &= 0 \\ ae^{\sqrt{-\lambda}L} - ae^{-\sqrt{-\lambda}L} &= 0 \\ \text{Multiply by } e^{\sqrt{-\lambda}L}\text{:} \quad ae^{2\sqrt{-\lambda}L} - a &= 0 \\ a\left(e^{2\sqrt{-\lambda}L} - 1\right) &= 0 \end{aligned}$$

Since $\lambda < 0$ and $L > 0$, we have $2\sqrt{-\lambda}L \neq 0$, so $e^{2\sqrt{-\lambda}L} \neq 1$. So we can divide by $e^{2\sqrt{-\lambda}L} - 1$ and get $a = 0$. But then $b = 0$, so once again we get $u(x,t) \equiv 0$.

Conclusion: This solution isn't productive. We must have $\lambda > 0$. Then we get complex solutions:

$$
\begin{aligned}
r &= \pm\sqrt{-\lambda} \\
&= \pm i\sqrt{\lambda} \\
X(x) &= a\cos\left(\sqrt{\lambda}x\right) + b\sin\left(\sqrt{\lambda}x\right) \\
u(x,t) &= \left[a\cos\left(\sqrt{\lambda}x\right) + b\sin\left(\sqrt{\lambda}x\right)\right] e^{-\lambda\alpha^2 t} \qquad \{\text{Absorb the }
\end{aligned}
$$

$$
\begin{aligned}
u(0,t) &= 0 \Longrightarrow a = 0 \\
u(x,t) &= b\sin\left(\sqrt{\lambda}x\right) e^{-\lambda\alpha^2 t} \\
u(L,t) &= 0 \Longrightarrow \\
b\sin\left(\sqrt{\lambda}L\right) e^{-\lambda\alpha^2 t} &= 0 \\
\text{Multiply by } e^{\lambda\alpha^2 t}: \quad b\sin\left(\sqrt{\lambda}L\right) &= 0 \quad \left\{\begin{array}{l}\text{If } b = 0\text{, we lose all solutions.} \\ \text{So we assume } b \neq 0\text{, so} \\ \sin\left(\sqrt{\lambda}L\right) = 0.\end{array}\right\} \\
\sqrt{\lambda}L &= n\pi \text{ for any integer } n \\
\lambda &= \frac{n^2\pi^2}{L^2} \\
\text{We get one solution for each } n: \quad X_n(x) &= b_n \sin\frac{n\pi x}{L} \\
T_n(t) &= e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \\
u_n(x,t) &= \boxed{b_n \sin\frac{n\pi x}{L} e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}}
\end{aligned}
$$