XXVI. Partial differential equations: Review of partial derivatives

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Lesson Overview

- Let $u(x,t)$ be a function of two variables.
- The partial derivative of u with respect to x is defined as

$$u_x = \frac{\partial u}{\partial x} := \lim_{h\to 0} \frac{u(x+h,t)-u(x,t)}{h}.$$

- Geometrically, u_x represents the slope as you walk in the x-direction on the surface.

Computing partial derivatives

- Algebraically, to find u_x you treat the other variable t as a constant and take the derivative with respect to x.
- We can also take second partial derivatives: $u_{xx}, u_{xt}, u_{tx}, u_{tt}$.
- Clairaut's Theorem says that the two "mixed partials" are always equal:

$$u_{xt} = u_{tx}$$

Example I

Let $u(x,t) = x^2 t$. Find the first partial derivatives u_x and u_t.

$u_x = \boxed{2xt}$. $u_t = \boxed{x^2}$.

Example II

Let $u(x,t) = \sin x \cos(t^2) + 3t$. Find the first partial derivatives u_x and u_t.

$u_x = \boxed{\cos x \cos t^2}$. $u_t = \boxed{-(\sin x) 2t \sin t^2 + 3}$.

Example III

Let $u(x,t) = x^2 + t^2$. Find all first and second partial derivatives and confirm Clairaut's Theorem for u.

$$\begin{aligned} u_x &= \boxed{2x} \\ u_t &= -\boxed{2t} \\ (u_x)_x &= -\boxed{2} \\ (u_x)_t &= \boxed{0} \\ (u_t)_t &= \boxed{2} \\ (u_t)_x &= \boxed{0} = (u_x)_t \checkmark \end{aligned}$$

Clairaut's Theorem holds!

Example IV

Let $u(x,t) = \frac{x}{x+t}$. Find all first and second partial derivatives and confirm Clairaut's Theorem for u.

$$
\begin{aligned}
u_x &= \boxed{\frac{t}{(x+t)^2}} \\
u_t &= -\boxed{\frac{x}{(x+t)^2}} \\
(u_x)_x &= -\boxed{\frac{2t}{(x+t)^3}} \\
(u_x)_t &= \frac{(x+t)^2 - 2t(x+t)}{(x+t)^4} = \boxed{\frac{x-t}{(x+t)^3}} \\
(u_t)_t &= \boxed{\frac{2x}{(x+t)^3}} \\
(u_t)_x &= -\frac{(x+t)^2 - 2x(x+t)}{(x+t)^4} = \boxed{\frac{x-t}{(x+t)^3}} = (u_x)_t \checkmark
\end{aligned}
$$

Clairaut's Theorem holds!

Example V

If $u_x = e^{xt} \cos t$, what is $u(x,t)$?

Treat t, and hence $\cos t$ as a constant:

$$
\begin{aligned}
u &= \int e^{xt} \cos t \, dx \\
&= \cos t \int e^{xt} \, dx \\
&= \frac{1}{t} (\cos t) e^{xt} + C
\end{aligned}
$$

But $u = \frac{1}{t}(\cos t)e^{xt} + t^2$ would also satisfy $u_x = e^{xt} \cos t$, since the t^2 would disappear under $\frac{\partial}{\partial x}$. So the general solution is

$$\boxed{u(x,t) = \frac{1}{t}(\cos t)e^{xt} + f(t)},$$

where $f(t)$ is any function of t only. (We don't have to include $+C$ since that could be included in $f(t)$.)