XXIV. Numerical techniques: Euler's method

Lesson Overview

- Euler's method is a way to find numerical approximations for initial value problems that we can't solve analytically.
- It is based on drawing lines along slopes in a direction field.

Formulas for Eulers method

- Start with an initial value problem in the form $y'(t) = f(t, y), y(t_0) = y_0$.
- Choose a step size h (usually given).
- Start at (t_0, y_0) and make iterative steps:

$$\begin{aligned} t_{n+1} &:= \boxed{t_n + h} \\ y_{n+1} &= \boxed{y_n + hf(t_n, y_n)} \end{aligned}$$

- Continue until you arrive at the value of t for which you need to approximate $y(t)$.

Example I

Use Euler's method with step size $h = 0.1$ to estimate $y(0.4)$ in the initial value problem $y' = 1 + t - y, y(0) = 1$.

$$\begin{aligned}
(t_0, y_0) = (0, 1) &\implies y' = f(0, 1) = 0 \\
y_1 &= 1 + 0.1(0) = 1 \\
(t_1, y_1) = (0.1, 1) &\implies y' = f(0.1, 1) = 0.1 \\
y_2 &= 1 + 0.1(0.1) = 1.01 \quad \{\text{Highlight this for use later. }\} \\
(t_2, y_2) = (0.2, 1.01) &\implies y' = f(0.2, 1.01) = 0.19 \\
y_3 &= 1.01 + 0.1(0.19) = 1.029 \\
(t_3, y_3) = (0.3, 1.029) &\implies y' = f(0.3, 1.029) = 0.271 \\
y_4 &= 1.029 + 0.1(0.271) = 1.029 + 0.0271 = 1.0561 \\
y(0.4) &\approx \boxed{1.0561}
\end{aligned}$$

Example II

Solve the initial value problem

$$y' = 1 + t - y, y(0) = 1$$

analytically. Compute $y(0.4)$ and compare the answer with the result given by Euler's method above.

$$\begin{aligned}
y' + y &= 1 + t \quad \{I(t) = e^t \quad \} \\
y'e^t + ye^t &= e^t + te^t \\
(ye^t)' &= e^t + te^t \\
ye^t &= te^t + C \\
y &= t + Ce^{-t} \\
1 &= 0 + C \\
C &= 1 \\
y &= \boxed{t + e^{-t}} \\
y(0.4) &= 0.4 + e^{-0.4} \approx \boxed{1.07032} \\
\text{Euler:} \quad y(0.4) &\approx \boxed{1.0561}
\end{aligned}$$

We were off by about 0.014. That's not bad.

Example III

Use Euler's method with step size $h = 0.1$ to estimate $y(0.4)$ in the initial value problem $y' = t^2 + y^2, y(0) = 1$.

$$\begin{aligned}
(0,1) &\rightarrow y' = 1 \\
(0.1,1.1) &\rightarrow y' = 0.01 + 1.21 = 1.22 \\
(0.2,1.222) &\rightarrow y' = 0.04 + 1.49328 = 1.53328 \\
(0.3,1.37533) &\rightarrow y' = 0.09 + 1.89153 = 1.98153 \\
(0.4,1.573) &
\end{aligned}$$

So $\boxed{y(0.4) \approx 1.573}$.

Example IV

Use Euler's method with step size $h = 0.3$ to estimate $y(0.6)$ in the initial value problem $y' = t - y, y(0) = 1$.

$$\begin{aligned}
(0,1) &\implies y' = -1 \\
(0.3,0.7) &\implies y' = 0.3 - 0.7 = -0.4 \\
(0.6,0.58) &
\end{aligned}$$

So $\boxed{y(0.4) \approx 0.58}$.

Example V

Solve the initial value problem

$$y' = t - y, y(0) = 1$$

analytically. Compute $y(0.6)$ and compare the answer with the result given by Euler's method above.

$$
\begin{aligned}
y' + y &= t \quad \{I(t) = e^t \quad\quad\quad\quad\quad\quad\quad\quad\quad \} \\
y'e^t + ye^t &= te^t \\
(ye^t)' &= te^t \\
ye^t &= te^t - e^t + C \\
y &= t - 1 + Ce^{-t} \\
y(0) = 1: \quad 1 &= -1 + C \\
C &= 2 \\
y &= \boxed{t - 1 + 2e^{-t}} \\
y(0.6) &= 0.6 - 1 + 2e^{-0.6} \approx \boxed{0.697623} \\
\text{Euler:} \quad y(0.6) &\approx \boxed{0.58}
\end{aligned}
$$