

XXIII. Inhomogeneous systems: variation of parameters

Lesson Overview

- We want to solve the inhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$.
- Find two solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$ using the methods of the previous lectures:

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix}$$

- Then look for a single particular solution \mathbf{x}_{par} to the inhomogeneous system as follows.

Solutions by variation of parameters

- Guess $\mathbf{x}_{\text{par}} = u_1(t)\mathbf{x}^{(1)} + u_2(t)\mathbf{x}^{(2)}$.
- Write the homogeneous solutions into a fundamental matrix:

$$\Psi := (\mathbf{x}^{(1)} \quad \mathbf{x}^{(2)}) = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix}$$

- Then $\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \Psi^{-1}\mathbf{g}(t)$.
- Integrate to get $u_1(t)$ and $u_2(t)$, and then plug into $\mathbf{x}_{\text{par}} = u_1(t)\mathbf{x}^{(1)} + u_2(t)\mathbf{x}^{(2)}$.

General solution and matrix inversion

- Add the homogeneous solution and particular solution to get the general solution:

$$\mathbf{x}_{\text{gen}} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + \mathbf{x}_{\text{par}}$$

- Hint for finding Ψ^{-1} :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example I

For the following inhomogeneous system, solve the corresponding homogeneous system and find the fundamental matrix:

$$\begin{aligned} x_1'(t) &= x_1(t) + x_2(t) + e^{-2t} \\ x_2'(t) &= 4x_1(t) - 2x_2(t) - 2e^t \end{aligned}$$

$$\begin{aligned} \mathbf{x}' &= \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} \implies r = 2, -3, \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \\ \mathbf{x}_{\text{hom}} &= \boxed{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-3t}} \\ \Psi &= \boxed{\begin{pmatrix} e^{2t} & -e^{-3t} \\ e^{2t} & 4e^{-3t} \end{pmatrix}} \end{aligned}$$

Example II

Solve the inhomogeneous system:

$$\begin{aligned}x_1'(t) &= x_1(t) + x_2(t) + e^{-2t} \\x_2'(t) &= 4x_1(t) - 2x_2(t) - 2e^t\end{aligned}$$

$$\begin{aligned}\mathbf{x}' &= \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} \implies r = 2, -3, \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \\ \mathbf{x}_{\text{hom}} &= \boxed{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-3t}} \\ \Psi &= \boxed{\begin{pmatrix} e^{2t} & -e^{-3t} \\ e^{2t} & 4e^{-3t} \end{pmatrix}} \quad \{\det = 5e^{-t} \quad \quad \quad \} \\ \Psi^{-1} &= \begin{pmatrix} \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} \\ -\frac{1}{5}e^{3t} & \frac{1}{5}e^{3t} \end{pmatrix} \\ \mathbf{g} &= \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix} \\ \mathbf{u}' &= \Psi^{-1} \mathbf{g}(t) = \begin{pmatrix} \frac{4}{5}e^{-4t} - \frac{2}{5}e^{-t} \\ -\frac{1}{5}e^t - \frac{2}{5}e^{4t} \end{pmatrix} \\ \mathbf{u} &= \begin{pmatrix} -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} \\ -\frac{1}{5}e^t - \frac{1}{10}e^{4t} \end{pmatrix}\end{aligned}$$

Example II

$$\begin{aligned}\Psi &= \begin{pmatrix} e^{2t} & -e^{-3t} \\ e^{2t} & 4e^{-3t} \end{pmatrix} \\ \mathbf{u} &= \begin{pmatrix} -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} \\ -\frac{1}{5}e^t - \frac{1}{10}e^{4t} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}
 \mathbf{x}_{\text{par}} &= \Psi \mathbf{u} \\
 &= \begin{pmatrix} e^{2t} & -e^{-3t} \\ e^{2t} & 4e^{-3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} \\ -\frac{1}{5}e^t - \frac{1}{10}e^{4t} \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + \frac{1}{5}e^{-2t} + \frac{1}{10}e^t \\ -\frac{1}{5}e^{-2t} + \frac{2}{5}e^t - \frac{4}{5}e^{-2t} - \frac{2}{5}e^t \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{2}e^t \\ -e^{-2t} \end{pmatrix} \left\{ \begin{array}{l} \text{Note uncanny amounts of} \\ \text{cancellation as is var of par's} \\ \text{wont.} \end{array} \right\} \\
 \mathbf{x}_{\text{gen}} &= \mathbf{x}_{\text{hom}} + \mathbf{x}_{\text{par}} \\
 &= \boxed{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}}
 \end{aligned}$$

Example III

Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

$$\begin{aligned}
 r &= \pm i \\
 r = i &\implies \mathbf{v} = \begin{pmatrix} 2 + i \\ 1 \end{pmatrix} \\
 r = -i &\implies \mathbf{v} = \begin{pmatrix} 2 - i \\ -1 \end{pmatrix}
 \end{aligned}$$

Example IV

For the following inhomogeneous system, solve the corresponding homogeneous system, find the fundamental matrix, and invert the matrix:

$$\begin{aligned}
 x_1'(t) &= 2x_1(t) - 5x_2(t) + \cos t \\
 x_2'(t) &= x_1(t) - 2x_2(t) + \sin t
 \end{aligned}$$

$$\begin{aligned}
 r = i &\implies \mathbf{v} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
 (\cos t + i \sin t) \begin{pmatrix} 2+i \\ 1 \end{pmatrix} &\implies \mathbf{x}_{\text{hom}} = c_1 \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix} \\
 \Psi &= \begin{pmatrix} 2 \cos t - \sin t & \cos t + 2 \sin t \\ \cos t & \sin t \end{pmatrix} \quad \{\det = -1\} \\
 \Psi^{-1} &= \begin{pmatrix} -\sin t & \cos t + 2 \sin t \\ \cos t & \sin t - 2 \cos t \end{pmatrix}
 \end{aligned}$$

Example V

Find a particular solution to the inhomogeneous system:

$$\begin{aligned}
 x_1'(t) &= 2x_1(t) - 5x_2(t) + \cos t \\
 x_2'(t) &= x_1(t) - 2x_2(t) + \sin t
 \end{aligned}$$

$$\begin{aligned}
 \Psi &= \begin{pmatrix} 2 \cos t - \sin t & \cos t + 2 \sin t \\ \cos t & \sin t \end{pmatrix} \quad \{\det = -1\} \\
 \mathbf{u}' &= \Psi^{-1} \mathbf{g} \\
 &= \begin{pmatrix} -\sin t & \cos t + 2 \sin t \\ \cos t & \sin t - 2 \cos t \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \\
 &= \begin{pmatrix} 2 \sin^2 t \\ 1 - 2 \sin t \cos t \end{pmatrix} \quad \left\{ \sin^2 t = \frac{1 - \cos 2t}{2} \right\} \\
 &= \begin{pmatrix} 1 - \cos 2t \\ 1 - 2 \sin t \cos t \end{pmatrix} \quad \left\{ \begin{array}{l} u := \cos t, du = -\sin t dt. \\ \text{[It's much uglier if you use]} \\ u := \sin t. \end{array} \right\} \\
 \mathbf{u} &= \begin{pmatrix} t - \frac{1}{2} \sin 2t \\ t + \cos^2 t \end{pmatrix} \\
 &= \begin{pmatrix} t - \sin t \cos t \\ t + \cos^2 t \end{pmatrix}
 \end{aligned}$$

Example V

$$\Psi = \begin{pmatrix} 2 \cos t - \sin t & \cos t + 2 \sin t \\ \cos t & \sin t \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} t - \sin t \cos t \\ t + \cos^2 t \end{pmatrix}$$

$$\begin{aligned} \mathbf{x}_{\text{par}} &= \Psi \mathbf{u} \\ &= \begin{pmatrix} 2 \cos t - \sin t & \cos t + 2 \sin t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} t - \sin t \cos t \\ t + \cos^2 t \end{pmatrix} \\ &= \begin{pmatrix} 2t \cos t - t \sin t - 2 \sin t \cos^2 t + \sin^2 t \cos t + t \cos t + 2t \sin t + \cos^3 t + 2 \sin t \cos^2 t \\ t \cos t - \sin t \cos^2 t + t \sin t + \sin t \cos^2 t \end{pmatrix} \\ &= \begin{pmatrix} 3t \cos t + t \sin t + [\sin^2 t \cos t + \cos^3 t] \\ t \cos t + t \sin t \end{pmatrix} \\ &= \begin{pmatrix} 3t \cos t + t \sin t + \cos t \\ t \cos t + t \sin t \end{pmatrix} \\ \mathbf{x}_{\text{par}} &= \boxed{t \begin{pmatrix} 3 \cos t + \sin t \\ \cos t + \sin t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t} \end{aligned}$$