XXII. Inhomogeneous systems: undetermined coefficients

Lesson Overview

- We want to solve the inhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$.
- First solve the corresponding homogeneous system $\mathbf{x}' = A\mathbf{x}$ using the methods of the previous lectures:

$$\mathbf{x}_{\text{hom}} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

Solving the inhomogeneous system

- Then look for a single particular solution $\mathbf{x}_{\text{par}}$ to the inhomogeneous system by guessing something of the same form as $\mathbf{g}(t)$ but with generic coefficients.
- Plug the guess into the system and solve for the coefficients.
- Add the homogeneous solution and the particular solution to get the general solution:

$$\mathbf{x}_{\text{hom}} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \mathbf{x}_{\text{par}}$$

Example I

For the following inhomogeneous system, solve the corresponding homogeneous system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t$$

$r = 3, -1$. $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$$\boxed{\mathbf{x}_{\text{hom}} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t}}$$

Example II

Solve the inhomogeneous system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t$$

$$\begin{aligned}
\mathbf{x}_{\text{hom}} &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} \\
\mathbf{x}_{\text{par}} &= \begin{pmatrix} a \\ b \end{pmatrix} e^t \\
\mathbf{x}'_{\text{par}} &= \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{x}_{\text{par}} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t \\
\begin{pmatrix} a \\ b \end{pmatrix} e^t &= \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^t + \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t \\
a &= a + 4b - 1 \Longrightarrow b = \frac{1}{4} \\
b &= a + b + 2 \Longrightarrow a = -2 \\
\mathbf{x}_{\text{par}} &= \begin{pmatrix} -2 \\ \frac{1}{4} \end{pmatrix} e^t \\
\mathbf{x}_{\text{gen}} &= \mathbf{x}_{\text{hom}} + \mathbf{x}_{\text{par}} = \boxed{c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} -2 \\ \frac{1}{4} \end{pmatrix} e^t}
\end{aligned}$$

Example III

For the following inhomogeneous system, solve the corresponding homogeneous system:

$$\mathbf{x}' = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^t \\ 5e^{3t} - 5e^t \end{pmatrix}$$

$r = 2, 8$. $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$$\boxed{\mathbf{x}_{\text{hom}} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{8t}}$$

Example IV

Find a partial particular solution to the following inhomogeneous system that accounts for the e^t terms on the right hand side:

$$\mathbf{x}' = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^t \\ 5e^{3t} - 5e^t \end{pmatrix}$$

$$\begin{aligned}
\mathbf{x}_{\text{par}} &= \begin{pmatrix} a \\ b \end{pmatrix} e^t \\
\mathbf{x}'_{\text{par}} &= \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \mathbf{x}_{\text{par}} + \begin{pmatrix} 2 \\ -5 \end{pmatrix} e^t \\
\begin{pmatrix} a \\ b \end{pmatrix} e^t &= \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^t + \begin{pmatrix} 2 \\ -5 \end{pmatrix} e^t \\
a &= 5a - 3b + 2 \Longrightarrow 4a - 3b = -2 \\
b &= -3a + 5b - 5 \Longrightarrow -3a + 4b = 5 \\
a &= 1 \\
b &= 2 \\
\mathbf{x}_{\text{par}} &= \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t}
\end{aligned}$$

Example V

Find a partial particular solution to the following inhomogeneous system that accounts for the e^{3t} terms on the right hand side:

$$\mathbf{x}' = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^t \\ 5e^{3t} - 5e^t \end{pmatrix}$$

$$\begin{aligned}
\mathbf{x}_{\text{par}} &= \begin{pmatrix} a \\ b \end{pmatrix} e^{3t} \\
\mathbf{x}'_{\text{par}} &= \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \mathbf{x}_{\text{par}} + \begin{pmatrix} 0 \\ 5 \end{pmatrix} e^{3t} \\
\begin{pmatrix} a \\ b \end{pmatrix} 3e^{3t} &= \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^{3t} + \begin{pmatrix} 0 \\ 5 \end{pmatrix} e^{3t} \\
3a &= 5a - 3b \Longrightarrow 2a - 3b = 0 \\
3b &= -3a + 5b - 5 \Longrightarrow -3a + 2b = -5 \\
a &= 3 \\
b &= 2 \\
\mathbf{x}_{\text{par}} &= \boxed{\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3t}}
\end{aligned}$$

Example VI

Find the general solution to the following inhomogeneous system:

$$\mathbf{x}' = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^t \\ 5e^{3t} - 5e^t \end{pmatrix}$$

$$\begin{aligned}
\mathbf{x}_{\text{gen}} &= \mathbf{x}_{\text{hom}} + \mathbf{x}_{\text{par}_1} + \mathbf{x}_{\text{par}_2} \\
&= \boxed{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{8t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t + \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3t}}
\end{aligned}$$