XXI. Systems of equations: repeated eigenvalues

Lesson Overview

- Recall that to solve the system of linear differential equations $\mathbf{x}' = A\mathbf{x}$, we find the eigenvalues and eigenvectors of A.
- If the characteristic equation has a repeated root, then we first find the corresponding eigenvector $\mathbf{v}$:

$$(A - rI)\mathbf{v} = \mathbf{0}$$

- Then find the generalized eigenvector $\mathbf{w}$:

$$(A - rI)\mathbf{w} = \mathbf{v}$$

Solutions from repeated eigenvalues

- Form the two principal solutions and the general solution:

$$\begin{aligned}
\mathbf{x}^{(1)} &= e^{rt}\mathbf{v} \quad \text{(as before)} \\
\mathbf{x}^{(2)} &= te^{rt}\mathbf{v} + e^{rt}\mathbf{w} \\
\mathbf{x}_{\text{gen}} &= c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} \\
\mathbf{x}_{\text{gen}} &= \boxed{c_1 e^{rt}\mathbf{v} + c_2\left(te^{rt}\mathbf{v} + e^{rt}\mathbf{w}\right)}
\end{aligned}$$

- Use initial conditions, if given, to solve for c_1 and c_2.

Graphing the solutions

- To graph the solutions, first graph $\mathbf{x}^{(1)}$ $(c_1 = 1, c_2 = 0)$, which is a straight line in the direction of $\mathbf{v}$.
- Other solutions $(c_2 \neq 0)$ swirl around this line.
- If $c_2 > 0$, then they approach positive multiples of $\mathbf{v}$.
- If $c_2 < 0$, then they approach negative multiples of $\mathbf{v}$.

Example I

Find the general solution to the following system:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \mathbf{x}$$

$r = 2, 2$. $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. $(A - rI)\mathbf{w} = \mathbf{v}$ gives $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Danger: You can't scale $\mathbf{w}$ by a constant (unless you scale $\mathbf{v}$ accordingly).

$$\boxed{\mathbf{x}_{\text{gen}} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \right]}$$

Example II

Graph some solution trajectories to the previous system of equations:

$$\mathbf{x}_{\text{gen}} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \right]$$

- $c_1 > 0, c_2 = 0 \implies$ axis. [Graph it pretty horozontal.]
- $c_1 < 0, c_2 = 0 \implies$ negative axis.
- $c_1 = 0, c_2 > 0 \implies$ starts on y-axis, but te^{2t} term pulls it towards upper right axis. (Label entire northwest half $\boxed{c_2 > 0}$, southeast half $\boxed{c_2 < 0}$.
- $c_1 = 0, c_2 < 0 \implies$ starts on negative y-axis, but te^{2t} term pulls it towards lower left axis.

Example III

Find the general solution to the following system:

$$\mathbf{x}' = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}$$

$r = -2, -2$. $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. $(A - rI)\mathbf{w} = \mathbf{v}$ gives $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\boxed{\mathbf{x}_{\text{gen}} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{-2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \right]}$$

Example IV

Graph some solution trajectories to the previous system of equations:

$$\mathbf{x}_{\text{gen}} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{-2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \right]$$

- $c_1 > 0, c_2 = 0 \implies$ axis.
- $c_1 < 0, c_2 = 0 \implies$ negative axis.
- $c_1 = 0, c_2 > 0 \implies$ starts on y-axis, but te^{-2t} term pulls it swirling in to 0 along the upper right axis.
- $c_1 = 0, c_2 < 0 \implies$ starts on negative y-axis, but te^{-2t} term pulls it swirling in to 0 along lower left axis.
- $c_1 = -1, c_2 =$ very small positive $\implies$ starts just above $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$, follows $c_1 < 0$ for a while, but then curls up and to the northeast to follow $c_1 > 0$.

Example V

Solve the following initial value problem:

$$\mathbf{x}' = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}, \mathbf{x}(0) = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{x}_{\text{gen}} &= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{-2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \right] \\
\mathbf{x}(0) &= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
1c_1 + 0c_2 &= 3 \implies c_1 = 3 \\
2x_1 + 1c_2 &= 8 \implies c_2 = 2 \\
\mathbf{x} &= 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + 2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{-2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \right] \\
&= \boxed{\begin{pmatrix} 3 \\ 8 \end{pmatrix} e^{-2t} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} te^{-2t}}
\end{aligned}$$