

XVII. Laplace transforms: initial value problems

Lesson Overview

- We start with an initial value problem:

$$ay'' + by' + cy = g(t), y(0) = y_0, y'(0) = y'_0$$

- We take the Laplace transform of both sides of the differential equation:

$$\begin{aligned}L\{ay'' + by' + cy\} &= L\{g(t)\} \\ aL\{y''\} + bL\{y'\} + cL\{y\} &= L\{g(t)\}\end{aligned}$$

- Now plug in the following identities:

$$\begin{aligned}L\{y''\} &= s^2L\{y\} - sy(0) - y'(0) \\ L\{y'\} &= sL\{y\} - y(0)\end{aligned}$$

- This gives us an equation that we can solve for $L\{y\}$ (in terms of s).
 - Then we take the inverse Laplace transform to find y .
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Example I

Solve the following initial value problem:

$$y'' + y' - 2y = 0, y(0) = 7, y'(0) = -2$$

$$\begin{aligned}L\{y''\} &= s^2L\{y\} - sy(0) - y'(0) = s^2L\{y\} - 7s + 2 \\L\{y'\} &= sL\{y\} - y(0) = sL\{y\} - 7 \\y'' + y' - 2y &= s^2L\{y\} - 7s + 2 + sL\{y\} - 7 = 0 \\(s^2 + s - 2)L\{y\} &= 7s + 5 \\L\{y\} &= \frac{7s + 5}{s^2 + s - 2} \\y &= \boxed{3e^{-2t} + 4e^t}\end{aligned}$$

Example II

Solve the following initial value problem:

$$y'' + 3y' = 0, y(0) = -2, y'(0) = 3$$

$$\begin{aligned}L\{y''\} &= s^2L\{y\} - sy(0) - y'(0) = s^2L\{y\} + 2s - 3 \\L\{y'\} &= sL\{y\} - y(0) = sL\{y\} + 2 \\y'' + 3y' &= s^2L\{y\} + 2s - 3 + 3sL\{y\} + 6 = 0 \\(s^2 + 3s)L\{y\} &= -2s - 3 \\L\{y\} &= \frac{-2s - 3}{s^2 + 3s} \\y &= \boxed{-1 - e^{-3t}}\end{aligned}$$

Example III

Solve the following initial value problem:

$$y'' + 4y' + 5y = 0, y(0) = 1, y'(0) = 0$$

$$\begin{aligned}
 L\{y''\} &= s^2L\{y\} - sy(0) - y'(0) = s^2L\{y\} - s \\
 L\{y'\} &= sL\{y\} - y(0) = sL\{y\} - 1 \\
 y'' + 4y' + 5y &= s^2L\{y\} - s + 4sL\{y\} - 4 + 5L\{y\} = 0 \\
 (s^2 + 4s + 5)L\{y\} &= s + 4 \\
 L\{y\} &= \frac{s + 4}{s^2 + 4s + 5} \\
 y &= \boxed{e^{-2t} \cos t + 2e^{-2t} \sin t}
 \end{aligned}$$

Example IV

Solve the following initial value problem:

$$y'' - 2y' + y = 4e^t, y(0) = 4, y'(0) = 1$$

$$\begin{aligned}
 L\{y''\} &= s^2L\{y\} - sy(0) - y'(0) = s^2L\{y\} - 4s - 1 \\
 L\{y'\} &= sL\{y\} - y(0) = sL\{y\} - 4 \\
 y'' - 2y' + y &= s^2L\{y\} - 4s - 1 - 2sL\{y\} + 8 + L\{y\} = L\{4e^t\} \\
 &= \frac{4}{s - 1} \\
 (s^2 - 2s + 1)L\{y\} &= \frac{4}{s - 1} + 4s - 7 = \frac{4 + (4s - 7)(s - 1)}{s - 1} \\
 &= \frac{4 + 4s^2 - 11s + 7}{s - 1} = \frac{4s^2 - 11s + 11}{s - 1} \\
 L\{y\} &= \frac{4s^2 - 11s + 11}{(s - 1)^3} \\
 y &= \boxed{4e^t - 3te^t + 2t^2e^t}
 \end{aligned}$$

Example V

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Solve the following initial value problem:

$$y'' + 4y = 8e^{-2t}, y(0) = 4, y'(0) = -4$$

$$L\{y''\} = s^2L\{y\} - sy(0) - y'(0) = s^2L\{y\} - 4s + 4$$

$$y'' + 4y = 8e^{-2t} \Rightarrow s^2L\{y\} - 4s + 4 + 4L\{y\} = L\{8e^{-2t}\} = \frac{8}{s+2}$$

$$(s^2 + 4)L\{y\} = \frac{8}{s+2} + 4s - 4 = \frac{8 + (4s - 4)(s + 2)}{s + 2}$$

$$= \frac{8 + 4s^2 + 4s - 8}{s + 2} = \frac{4s^2 + 4s}{s + 2}$$

$$L\{y\} = \frac{4s^2 + 4s}{(s + 2)(s^2 + 4)}$$

$$y = \boxed{e^{-2t} + 3 \cos 2t - \sin 2t}$$