XVII. Laplace transforms: initial value problems

Lesson Overview

- We start with an initial value problem:

$$ay'' + by' + cy = g(t), y(0) = y_0, y'(0) = y'_0$$

- We take the Laplace transform of both sides of the differential equation:

$$\begin{aligned} L\{ay'' + by' + cy\} &= L\{g(t)\} \\ aL\{y''\} + bL\{y'\} + cL\{y\} &= L\{g(t)\} \end{aligned}$$

- Now plug in the following identities:

$$\begin{aligned} L\{y''\} &= s^2 L\{y\} - sy(0) - y'(0) \\ L\{y'\} &= sL\{y\} - y(0) \end{aligned}$$

- This gives us an equation that we can solve for $L\{y\}$ (in terms of s).
- Then we take the inverse Laplace transform to find y.

Example I

Solve the following initial value problem:

$$y'' + y' - 2y = 0, y(0) = 7, y'(0) = -2$$

$$
\begin{aligned}
L\{y''\} &= s^2 L\{y\} - sy(0) - y'(0) = s^2 L\{y\} - 7s + 2 \\
L\{y'\} &= sL\{y\} - y(0) = sL\{y\} - 7 \\
y'' + y' - 2y &= s^2 L\{y\} - 7s + 2 + sL\{y\} - 7 = 0 \\
\left(s^2 + s - 2\right) L\{y\} &= 7s + 5 \\
L\{y\} &= \frac{7s + 5}{s^2 + s - 2} \\
y &= \boxed{3e^{-2t} + 4e^t}
\end{aligned}
$$

Example II

Solve the following initial value problem:

$$y'' + 3y' = 0, y(0) = -2, y'(0) = 3$$

$$
\begin{aligned}
L\{y''\} &= s^2 L\{y\} - sy(0) - y'(0) = s^2 L\{y\} + 2s - 3 \\
L\{y'\} &= sL\{y\} - y(0) = sL\{y\} + 2 \\
y'' + 3y' &= s^2 L\{y\} + 2s - 3 + 3sL\{y\} + 6 = 0 \\
\left(s^2 + 3s\right) L\{y\} &= -2s - 3 \\
L\{y\} &= \frac{-2s - 3}{s^2 + 3s} \\
y &= \boxed{-1 - e^{-3t}}
\end{aligned}
$$

Example III

Solve the following initial value problem:

$$y'' + 4y' + 5y = 0, y(0) = 1, y'(0) = 0$$

$$
\begin{aligned}
L\{y''\} &= s^2L\{y\} - sy(0) - y'(0) = s^2L\{y\} - s \\
L\{y'\} &= sL\{y\} - y(0) = sL\{y\} - 1 \\
y'' + 4y' + 5y &= s^2L\{y\} - s + 4sL\{y\} - 4 + 5L\{y\} = 0 \\
\left(s^2 + 4s + 5\right) L\{y\} &= s + 4 \\
L\{y\} &= \frac{s+4}{s^2 + 4s + 5} \\
y &= \boxed{e^{-2t}\cos t + 2e^{-2t}\sin t}
\end{aligned}
$$

Example IV

Solve the following initial value problem:

$$y'' - 2y' + y = 4e^t, y(0) = 4, y'(0) = 1$$

$$
\begin{aligned}
L\{y''\} &= s^2L\{y\} - sy(0) - y'(0) = s^2L\{y\} - 4s - 1 \\
L\{y'\} &= sL\{y\} - y(0) = sL\{y\} - 4 \\
y'' - 2y' + y &= s^2L\{y\} - 4s - 1 - 2sL\{y\} + 8 + L\{y\} = L\left\{4e^t\right\} \\
&= \frac{4}{s-1} \\
\left(s^2 - 2s + 1\right) L\{y\} &= \frac{4}{s-1} + 4s - 7 = \frac{4 + (4s-7)(s-1)}{s-1} \\
&= \frac{4 + 4s^2 - 11s + 7}{s-1} = \frac{4s^2 - 11s + 11}{s-1} \\
L\{y\} &= \frac{4s^2 - 11s + 11}{(s-1)^3} \\
y &= \boxed{4e^t - 3te^t + 2t^2e^t}
\end{aligned}
$$

Example V

Solve the following initial value problem:

$$y'' + 4y = 8e^{-2t}, y(0) = 4, y'(0) = -4$$

$$\begin{aligned}
L\left\{y''\right\} &= s^2 L\left\{y\right\} - sy(0) - y'(0) = s^2 L\left\{y\right\} - 4s + 4 \\
y'' + 4y &= s^2 L\left\{y\right\} - 4s + 4 + 4L\left\{y\right\} = L\left\{8e^{-2t}\right\} = \frac{8}{s+2} \\
\left(s^2 + 4\right) L\left\{y\right\} &= \frac{8}{s+2} + 4s - 4 = \frac{8 + (4s-4)(s+2)}{s+2} \\
&= \frac{8 + 4s^2 + 4s - 8}{s+2} = \frac{4s^2 + 4s}{s+2} \\
L\left\{y\right\} &= \frac{4s^2 + 4s}{(s+2)(s^2+4)} \\
y &= \boxed{e^{-2t} + 3\cos 2t - \sin 2t}
\end{aligned}$$