XV. Laplace transforms

Lesson Overview

- The Laplace transform of a function $f(t)$ is
$$L\{f\} := \int_0^\infty e^{-st} f(t)\,dt.$$
- Note that it will be a function of s.
- The Laplace transform is linear, meaning that for functions $f(t)$ and $g(t)$ and constants a and b, we have
$$L\{af + bg\} = aL\{f\} + bL\{g\}.$$

Example I

Find the Laplace transform of $f(t) = t^n, n \geq 0$.

$$\begin{aligned}
L\{1\} &:= \int_0^\infty e^{-st}\,dt \\
&= -\frac{1}{s}e^{-st}\Big|_{t=0}^{t=\infty} \\
&= \boxed{\frac{1}{s}} \\
L\{t\} &:= \int_0^\infty te^{-st}\,dt \qquad \{\text{Integrate by parts.} \qquad\qquad \} \\
&= \left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_{t=0}^{t=\infty} \\
&= \boxed{\frac{1}{s^2}} \\
L\left\{t^2\right\} &:= \int_0^\infty t^2e^{-st}\,dt \qquad \{\text{Integrate by parts.} \qquad\qquad \} \\
&= \left(-\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right)\Big|_{t=0}^{t=\infty} \\
&= \boxed{\frac{2}{s^3}} \\
L\left\{t^3\right\} &:= \int_0^\infty t^2e^{-st}\,dt \qquad \{\text{Integrate by parts.} \qquad\qquad \} \\
&= \left(-\frac{t^3}{s}e^{-st} - \frac{3t^2}{s^2}e^{-st} - \frac{6t}{s^3}e^{-st} - \frac{6}{s^4}e^{-st}\right)\Big|_{t=0}^{t=\infty} \\
&= \boxed{\frac{6}{s^4}} \\
L\{t^n\} &= \boxed{\frac{n!}{s^{n+1}}}
\end{aligned}$$

Example II

Find the Laplace transform of $f(t) = e^{at}$, assuming $s > a$.

$$\begin{aligned} s > a &\implies s - a > 0 \\ L\left\{e^{at}\right\} &:= \int_0^\infty e^{-st} e^{at}\, dt \\ &= \int_0^\infty e^{-(s-a)t}\, dt \\ &= -\frac{e^{-(s-a)t}}{s-a}\Bigg|_{t=0}^{t=\infty} \\ &= \boxed{\frac{1}{s-a}} \end{aligned}$$

Example III

Find the Laplace transform of $f(t) = \cos at$.

$$\begin{aligned} L\left\{\cos at\right\} &:= \int_0^\infty e^{-st} \cos at\, dt && \left\{\text{Integrate by parts twice, or use a CAS or an integral table.}\right\} \\ &= \left(-\frac{s}{a^2+s^2} e^{-st} \cos at + \frac{a}{a^2+s^2} e^{-st} \sin at\right)\Bigg|_{t=0}^{t=\infty} && \left\{\text{The } \infty \text{ terms go to 0, and } \sin 0 = 0.\right\} \\ &= \boxed{\frac{s}{a^2+s^2}} \end{aligned}$$

Example IV

Find the Laplace transform of $f(t) = \sin at$.

$$\begin{aligned}
L\{\sin at\} &:= \int_0^\infty e^{-st}\sin at\,dt && \left\{\begin{array}{l}\text{Integrate by parts twice, or}\\ \text{use a CAS or an integral}\\ \text{table.}\end{array}\right\}\\
&= \left(-\frac{a}{a^2+s^2}e^{-st}\cos at + \frac{s}{a^2+s^2}e^{-st}\sin at\right)\Bigg|_{t=0}^{t=\infty} && \left\{\begin{array}{l}\text{The } \infty \text{ terms go to 0, and}\\ \sin 0 = 0.\end{array}\right\}\\
&= \boxed{\frac{a}{a^2+s^2}}
\end{aligned}$$

Example V

Find the Laplace transform of the following function:

$$f(t) = 3\cos 4t - 2\sin 5t + e^{2t} + 3t^2 + 7t - 2$$

Use linearity:

$$\begin{aligned}
L\{1\} &= \frac{1}{s}\\
L\{t\} &= \frac{1}{s^2}\\
L\left\{t^2\right\} &= \frac{2}{s^3}\\
L\left\{e^{at}\right\} &= \frac{1}{s-a}\\
L\{\cos at\} &= \frac{s}{a^2+s^2}\\
L\{\sin at\} &= \frac{a}{a^2+s^2}\\
L\{f\} &= \boxed{\frac{3s}{s^2+16} - \frac{10}{s^2+25} + \frac{1}{s-2} + \frac{6}{s^3} + \frac{7}{s^2} - \frac{2}{s}}
\end{aligned}$$