XI. Review of power series

Lesson Overview

- A function $f(x)$ has a Taylor Series expansion around a point x_0:

  $$TS(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \text{ where } a_n = \frac{f^{(n)}(x_0)}{n!}$$

  If $x_0 = 0$, it's also called Maclaurin Series.

- Common Maclaurin Series to remember from calculus:

  $$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \text{ for } |x| < 1 \end{aligned}$$

Any power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a radius of convergence R around x_0, i.e. when you plug in values for x satisfying $|x-x_0|<R$, it converges. It might or might not converge at the endpoints $x=x_0-R, x=x_0+R$.

Extreme cases:

- $R=0$. It only converges for $x=x_0$.
- $R=\infty$. Then it converges for all $x\in\mathbb{R}$.
- To find the radius of convergence, we usually use the Ratio Test: A series converges if

$$\lim_{n\to\infty}\left|\frac{\text{term}_{n+1}}{\text{term}_n}\right|<1.$$

- We must check the endpoints separately (using a non-Ratio test).

Example I

Identify the following power series as an elementary function:

$$1+3x^2+\frac{9}{2}x^4+\frac{9}{2}x^6+\frac{27}{8}x^8+\cdots$$

$$\begin{aligned}
&= 1+3x^2+\frac{9}{2}x^4+\frac{27}{6}x^6+\frac{81}{24}x^8+\cdots\\
&= 1+3x^2+\frac{(3x^2)^2}{2!}+\frac{(3x^2)^3}{6}+\frac{(3x^2)^4}{24}+\cdots\\
&= \boxed{e^{3x^2}}
\end{aligned}$$

Example II

Find the Maclaurin Series for $f(x) = \ln(1-x)$.

Lesson from Calc II: Writing down $f(x), f'(x), f''(x), \ldots$ is usually the worst way to find a Taylor Series.

Instead, note that

$$\begin{aligned}\ln(1-x) &= -\int \frac{dx}{1-x} \\ &= -\int (1 + x + x^2 + x^3 + \cdots)\, dx \\ &= C - x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\end{aligned}$$

Plug in $x = 0$ to get $C = 0$.

So $a_n = 0$ for $n = 0, a_n = -\frac{1}{n}$ for $n = 1, 2, 3, \ldots$.

$$\ln(1-x) = \boxed{\sum_{n=1}^{\infty} \left(-\frac{x^n}{n}\right)}$$

Example III

Find the interval of convergence for the Maclaurin Series for $\ln(1-x)$.

Use the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{-\frac{x^{n+1}}{n+1}}{-\frac{x^n}{n}} \right| = |x| < 1$$

So it converges for $|x| < 1$. Ratio $R = 1$ around $x_0 = 0$.

Check the endpoints separately (using a non-Ratio test): $x = 1$ gives $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$, which diverges (Harmonic Series/p-series). $x = -1$ gives $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, which converges (Alternating Series Test).

So this series converges for $\boxed{-1 \leq x < 1}$, or $\boxed{[-1, 1)}$. This was predictable since $\ln(1-x)$ blows up at $x = 1$.

Example IV

Use power series to solve the following integral:

$$\int e^{x^2}\, dx$$

$\int e^{x^2}\, dx$ can not be done by any integration technique you learned in Calc II (substitution, parts, partial fractions, etc.). That's because there is no "elementary function" whose derivative is e^{x^2}. But we can find a series that works:

$$\begin{aligned} \int e^{x^2}\, dx &= \int \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) dx \\ &= \boxed{C + x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \cdots} \end{aligned}$$

Alternately,

$$\int \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right) dx = \boxed{C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}}.$$

Example V

Suppose $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Find power series expressions for $y'(x)$ and $y''(x)$ and shift the indices of summation so that they start at $n = 0$.

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} && \left\{\begin{array}{l}\text{Omit the } n = 0 \text{ term} \\ \text{because it is 0 anyway.}\end{array}\right\} \\
&= a_1 + 2a_2 x + 3a_3 x^2 + \cdots && \left\{\begin{array}{l}\text{Shift the index of} \\ \text{summation by 1.}\end{array}\right\} \\
&= \boxed{\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1}}
\end{aligned}
$$

$$
\left\{\begin{array}{l}\textbf{Mnemonic}\text{: If you lower} \\ \text{the } n \text{ in the } \underline{\text{index}} \text{ by 1, then} \\ \text{raise the } n\text{'s in the } \underline{\text{formula}} \\ \text{by 1.}\end{array}\right\}
$$

$$
\begin{aligned}
y'' &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} && \left\{\begin{array}{l}\text{Omit the } n = 0 \text{ and } n = 1 \\ \text{terms because they are 0.}\end{array}\right\} \\
&= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \boxed{\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n} && \{\text{Shifting } n \text{ by 2.}\}
\end{aligned}
$$