

X. Inhomogeneous equations: variation of parameters

Lesson Overview

- To solve the (linear, second-order, inhomogeneous, constant coefficient) differential equation

$$y'' + by' + cy = g(t)$$

first solve the homogeneous equation

$$y'' + by' + cy = 0$$

by the methods of earlier lectures to get

$$y_{\text{hom}} = c_1y_1 + c_2y_2.$$

- Then find a particular solution to the inhomogeneous equation

$$y'' + by' + cy = g(t)$$

using variation of parameters. This means you guess

$$y_{\text{par}} = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where $u_1(t)$ and $u_2(t)$ are solutions to the system:

$$\begin{aligned}y_1u_1' + y_2u_2' &= 0 \\y_1'u_1 + y_2'u_2 &= g\end{aligned}$$

- Solve the system for u_1' and u_2' .
- Integrate to get u_1 and u_2 .

- Plug in to $y_{\text{par}} = u_1 y_1 + u_2 y_2$.
- Hint on solving the system:
- Find the Wronskian of the two homogeneous solutions:

$$W(y_1, y_2) := \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}, |W| = y_1 y_2' - y_2 y_1'$$

- Then you can find u_1' and u_2' :

$$u_1' = -\frac{y_2 g}{|W|}$$
$$u_2' = \frac{y_1 g}{|W|}$$

Example I

The homogeneous differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 0$$

has general solution

$$y_{\text{hom}} = c_1 t + c_2 t e^t.$$

Find the general solution to the inhomogeneous differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 2t^3.$$

Example I

$$t^2 y'' - t(t+2)y' + (t+2)y = 2t^3, y_{\text{hom}} = c_1 t + c_2 t e^t$$

$$\begin{aligned}
 g(t) &= \frac{2t^3}{t^2} = 2t \\
 |W(y_1, y_2)| &= \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2e^t + te^t - te^t = t^2e^t \\
 u_1' &= -\frac{gy_2}{|W|} = \frac{-2t \cdot te^t}{t^2e^t} = -2 \\
 u_1 &= -2t \\
 u_2' &= \frac{gy_1}{|W|} = \frac{2t \cdot t}{t^2e^t} = 2e^{-t} \\
 u_2 &= -2e^{-t} \\
 y_{\text{par}} &= u_1y_1 + u_2y_2 = -2t \cdot t - 2e^{-t} \cdot te^t = -2t^2 - 2t
 \end{aligned}$$

Note that $-2t$ is a multiple of the homogeneous solution y_2 , so we can omit it:

$$y_{\text{gen}} = c_1t + c_2te^t - 2t^2$$

Example II

For the inhomogeneous differential equation

$$y'' + y = \tan t$$

solve the corresponding homogeneous equation and find the Wronskian of the solutions.

$$\begin{aligned}
 y_{\text{hom}} &= c_1 \sin t + c_2 \cos t \\
 y_1 = \sin t, y_2 = \cos t &\implies |W| = -1
 \end{aligned}$$

Note: Normally, $|W|$ will be a function of t , not a number. We just got lucky here.

Example III

Solve the inhomogeneous differential equation

$$y'' + y = \tan t.$$

$$\begin{aligned}
 y_{\text{hom}} &= c_1 \sin t + c_2 \cos t \\
 (\sin t)u'_1 + (\cos t)u'_2 &= 0 \quad \{(*)\} \\
 (\cos t)u'_1 - (\sin t)u'_2 &= \tan t \quad \{(**)\} \\
 y_1 = \sin t, y_2 = \cos t, g = \tan t &\implies |W| = -1 \\
 u'_1 &= \frac{-y_2 g}{|W|} \\
 u'_2 &= \frac{y_1 g}{|W|} \\
 u'_1 &= \cos t \tan t = \sin t, \text{ so } u_1 = -\cos t \\
 u'_2 &= -\sin t \tan t = -\frac{\sin^2 t}{\cos t} = \frac{\cos^2 t - 1}{\cos t} = \cos t - \sec t \\
 u_2 &= \sin t - \ln(\sec t + \tan t) \\
 y_{\text{gen}} &= c_1 \sin t + c_2 \cos t - \sin t \cos t + \sin t \cos t - \cos t \ln(\sec t + \tan t) \\
 &= \boxed{c_1 \sin t + c_2 \cos t - \cos t \ln(\sec t + \tan t)}
 \end{aligned}$$

Example IV

For the inhomogeneous differential equation

$$y'' + 3y' + 2y = \sin e^t$$

solve the corresponding homogeneous equation and find the Wronskian of the solutions.

$$r = -1, -2 \implies \boxed{y_{\text{hom}} = c_1 e^{-t} + c_2 e^{-2t}}$$

$$W(y_1, y_2) := \begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix}, |W| = -2e^{-3t} + e^{-3t} = \boxed{-e^{-3t}}$$

Example V

Solve the inhomogeneous differential equation

$$y'' + 3y' + 2y = \sin e^t.$$

$$u_1' = -\frac{y_2 g}{|W|} = \frac{-e^{-2t} \sin e^t}{-e^{-3t}} = e^t \sin e^t \quad \left\{ \begin{array}{l} \text{Integrate using } s = e^t, ds = \\ e^t dt. \end{array} \right\}$$

$$u_1 = -\cos e^t$$

$$u_2' = \frac{y_1 g}{|W|} = \frac{e^{-t} \sin e^t}{-e^{-3t}} = -e^{2t} \sin e^t \quad \left\{ \begin{array}{l} \text{Integrate using } s = e^t, ds = \\ e^t dt. \end{array} \right\}$$

$$-\int s \sin s ds = -(-s \cos s + \sin s) \quad \{ \text{No } +C \text{ necessary.} \quad \}$$

$$u_2 = e^t \cos e^t - \sin e^t$$

Plug in:

$$y_{\text{par}} = -e^{-t} \cos e^t + e^{-t} \cos e^t - e^{-2t} \sin e^t$$

$$= -e^{-2t} \sin e^t$$

$$y_{\text{gen}} = \boxed{c_1 e^{-t} + c_2 e^{-2t} - e^{-2t} \sin e^t}$$