I. Linear equations

Lesson Overview

- Today we'll learn how to solve linear differential equations:

$$y'(x) + P(x)y(x) = Q(x)$$

- **Notes on the form**:
  1. Linear means that we think of y and y' as the variables (not x and y). We think of as $P(x)$ and $Q(x)$ as coefficients: The equation has the form $y' + Py = Q$, which would be a line.
  2. If there is a coefficient in front of $y'(x)$, make sure you divide it away before using the algorithm below.

How to solve linear equations

$$y'(x) + P(x)y(x) = Q(x)$$

1. Calculate the integrating factor

$$I(x) := e^{\int P(x)\,dx}$$

   and multiply that by both sides.

2. This makes the left hand side into

$$e^{\int P(x)\,dx}y' + P(x)e^{\int P(x)\,dx}y = Iy' + I'y = (Iy)',$$

   so we can then integrate both sides.

Solving linear equations

$$y'(x) + P(x)y(x) = Q(x)$$

3. Then you'll get

$$I(x)y(x) = \int I(x)Q(x)\,dx + C$$

and you can solve for $y(x)$.

Further notes

- If $P(x)$ is negative, make sure to include that in finding $I(x)$. And remember that $e^{-\ln(\text{cucumber})}$ doesn't simplify to $-(\text{cucumber})$! It's $\frac{1}{(\text{cucumber})}$.
- When you're integrating $P(x)$ to find the integrating factor $I(x)$, it's ok to leave off the constant C.
- However, when you're integrating both sides of the equation, the C is very important. And it's important that you add it when you do the integration and keep track of it in the ensuing algebra.

Example I

Find the general solution to the following differential equation:

$$y' + xy = x^3$$

Example I

$$y'(x) + xy = x^3$$

Multiply both sides by $e^{\frac{x^2}{2}}$:

$$e^{\frac{x^2}{2}}y' + xe^{\frac{x^2}{2}}y = x^3e^{\frac{x^2}{2}}$$

Point: The LHS is now $\left(ye^{\frac{x^2}{2}}\right)'$, using the Product Rule.

$$\left(ye^{\frac{x^2}{2}}\right)' = x^3e^{\frac{x^2}{2}} \quad \{\text{Integrate both sides:} \qquad \}$$

RHS: $u := \frac{x^2}{2}$ $\qquad du = x\,dx$

$$\begin{aligned}
\int x^3e^{\frac{x^2}{2}}\,dx &= \int x^2e^{\frac{x^2}{2}}x\,dx \\
&= \int 2ue^u\,du \\
\text{Use parts:} \quad &= 2\left(ue^u - e^u\right) + C = 2\left(\frac{x^2}{2}e^{\frac{x^2}{2}} - e^{\frac{x^2}{2}}\right) + C = x^2e^{\frac{x^2}{2}} - 2e^{\frac{x^2}{2}} + C \\
ye^{\frac{x^2}{2}} &= (x^2-2)e^{\frac{x^2}{2}} + C \\
y &= \boxed{x^2 - 2 + Ce^{-\frac{x^2}{2}}} \quad \{(\underline{\text{Not}}\ y = x^2 - 2 + C!) \qquad \}
\end{aligned}$$

Now use IC (if given) to get C.

Example II

Solve the following initial value problem:

$$(\cos x)y' + (\sin x)y = \cos^5 x \sin x, y(0) = 2$$

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$$\begin{aligned}
(\cos x)y' + (\sin x)y &= \cos^5 x \sin x, y(0) = 2 \\
y' + (\tan x)y &= \cos^4 x \sin x \\
I(x) &= e^{\int \tan x\, dx} \\
&= e^{-\ln \cos x} \\
&= \sec x \\
(\sec x)y' + (\sec x \tan x)y &= \cos^3 x \sin x \quad \left\{\text{At this point, check* whether the LHS really is the derivative of } (\sec x)y.\right\} \\
(\sec x)y &= -\frac{1}{4}\cos^4 x + C \\
y &= C\cos x - \frac{1}{4}\cos^5 x \\
y(0) &= C - \frac{1}{4} = 2 \\
C &= \frac{9}{4} \\
y &= \boxed{\frac{9}{4}\cos x - \frac{1}{4}\cos^5 x}
\end{aligned}$$

Example III

For the linear differential equation

$$(x+1)y' - y = \sin x,$$

what is $I(x)$?

$$\boxed{\frac{1}{x+1}}$$

Example IV

For the linear differential equation

$$(\sin x)y' - (\cos x)y = x^2,$$

what is $I(x)$?

$\boxed{\csc x}$

Example V

Solve the initial value problem:

$$(t+1)y' - 3y = t, y(1) = 2$$

Example V

$$(t+1)y' - 3y = t, y(1) = 2$$

$$
\begin{aligned}
y' - \frac{3}{t+1}y &= \frac{t}{t+1} \\
I(t) &:= e^{-\int \frac{3}{t+1}\,dt} \\
&= e^{-3\ln(t+1)} \\
&= \frac{1}{(t+1)^3} \\
\frac{y'}{(t+1)^3} - \frac{3}{(t+1)^4}y &= \frac{t}{(t+1)^4} \\
\left(\frac{y}{(t+1)^3}\right)' &= \frac{t}{(t+1)^4} \\
\frac{y}{(t+1)^3} &= \int \frac{t}{(t+1)^4}\,dt \qquad \{u := t+1, du = dt \qquad\qquad \} \\
&= \int \frac{u-1}{u^4}\,du \\
&= \int \left(\frac{1}{u^3} - \frac{1}{u^4}\right) du \\
&= -\frac{1}{2(t+1)^2} + \frac{1}{3(t+1)^3} + C \\
y &= -\frac{1}{2}(t+1) + \frac{1}{3} + C(t+1)^3 \\
2 &= -\frac{1}{2}(2) + \frac{1}{3} + 8C \\
\frac{8}{3} &= 8C \\
C &= \frac{1}{3} \\
y &= \boxed{\frac{1}{3}\left[(t+1)^3 + 1\right] - \frac{1}{2}(t+1)} \\
&= \frac{1}{3}t^3 + t^2 + \frac{1}{2}t + \frac{1}{6}
\end{aligned}
$$

Example VI

Find the general solution to the following differential equation:

$$xy' + 3y = \cos x, x > 0$$

Example VI

$$\begin{aligned} y' + \frac{3}{x}y &= \frac{\cos x}{x} \\ I(x) &= e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = e^{x^3} = x^3 \\ x^3y' + 3x^2y &= x^2\cos x \end{aligned}$$

$$\begin{array}{r|l} x^2 & \cos x \\ \hline 2x & \sin x \\ 2 & -\cos x \\ 0 & -\sin x \end{array}$$

$$\begin{aligned} x^3y &= x^2\sin x + 2x\cos x - 2\sin x + C \\ y &= \boxed{\frac{x^2\sin x + 2x\cos x - 2\sin x + C}{x^3}} \end{aligned}$$