XVII. Density and Cumulative Distribution Functions

Density Functions

- Let Y be a continuous random variable. It has a density function $f(y)$ that satisfies
  1. $f(y) \geq 0$, and
  2. $\int_{-\infty}^{\infty} f(y)\,dy = 1$.
- Use the density function to calculate probabilities:

$$P(a \leq Y \leq b) = \boxed{\int_a^b f(y)\,dy}$$

Cumulative Distribution Functions

- If Y has density function f, then it has cumulative distribution function

$$F(y) := P(Y \leq y) = \int_{-\infty}^{y} f(t)\,dt.$$

- We can also use F to calculate probabilities:

$$P(a \leq Y \leq b) = \int_a^b f(y)\,dy = \boxed{F(b) - F(a)}$$

Properties of the CDF

1. $F(-\infty) = 0$.

2. $F(\infty) = 1$.
3. F is increasing.
4. $F'(y) = f(y)$.

Graph the typical s-curve.

Example I

Let Y have density function

$$f(y) := \begin{cases} cy, & 0 \leq y \leq 2, \\ c(4-y), & 2 < y \leq 4, \\ 0, & \text{elsewhere.} \end{cases}$$

A. Find c.

B. Find $F(y)$, the cumulative distribution function of Y.

Example I

$$f(y) := \begin{cases} cy, & 0 \leq y \leq 2, \\ c(4-y), & 2 < y \leq 4, \\ 0, & \text{elsewhere.} \end{cases}$$

A.

$$
\begin{aligned}
\int_0^4 f(y)\,dy &= c\left(\int_0^2 y\,dy + \int_2^4 (4-y)\,dy\right) \\
&= c\left[\left.\frac{y^2}{2}\right|_{y=0}^{y=2} + \left.\left(4y-\frac{y^2}{2}\right)\right|_{y=2}^{y=4}\right] \\
&= c\,(2+16-8-8+2) \\
&= 4c = 1 \\
c &= \boxed{\frac{1}{4}}
\end{aligned}
$$

B.

$$
\begin{aligned}
F(y) &= \int_0^y f(t)\,dt \\
&= \begin{cases} \frac{1}{4}\int_0^y t\,dt, & 0 \le y \le 2, \\ \frac{1}{4}\left(\int_0^2 t\,dt + \int_2^y (4-t)\,dt\right), & 2 < y \le 4 \end{cases} \\
&= \begin{cases} \frac{y^2}{8} & 0 \le y \le 2, \\ \frac{1}{4}\left[2 + \left.\left(4t-\frac{t^2}{2}\right)\right|_{t=2}^{t=y}\right], & 2 < y \le 4 \end{cases} \\
&= \begin{cases} \frac{y^2}{8} & 0 \le y \le 2, \\ \frac{1}{4}\left(2 + 4y - \frac{y^2}{2} - 8 + 2\right), & 2 < y \le 4 \end{cases} \\
&= \boxed{\begin{cases} 0, & y < 0, \\ \frac{y^2}{8} & 0 \le y \le 2, \\ -\frac{y^2}{8} + y - 1, & 2 < y \le 4, \\ 1, & 4 < y \end{cases}}
\end{aligned}
$$

As a check, we get $F(2) = \frac{1}{2}$ using either part of the function, and $F(4) = 1$.

Example II

As in Example I, let Y have density function

$$f(y) := \begin{cases} \frac{1}{4}y, & 0 \le y \le 2, \\ \frac{1}{4}(4-y), & 2 < y \le 4, \\ 0, & \text{elsewhere.} \end{cases}$$

A. Find $P(1 \le Y \le 3)$.

B. Find $P(Y \le 2 | Y \ge 1)$.

Example II

$$f(y) := \begin{cases} \frac{1}{4}y, & 0 \le y \le 2, \\ \frac{1}{4}(4-y), & 2 < y \le 4 \end{cases}$$

B. Find $P(Y \le 2 | Y \ge 1)$.

$$F(y) = \begin{cases} 0, & y < 0, \\ \dfrac{y^2}{8} & 0 \le y \le 2, \\ -\dfrac{y^2}{8} + y - 1, & 2 < y \le 4, \\ 1, & 4 < y \end{cases}$$

A.

$$\begin{aligned} P(1 \le Y \le 3) &= F(3) - F(1) \\ &= -\frac{9}{8} + 3 - 1 - \frac{1}{8} \\ &= \boxed{\frac{3}{4}} \end{aligned}$$

B.

$$\begin{aligned} P(Y \le 2 | Y \ge 1) &= \frac{P(1 \le Y \le 2)}{P(Y \ge 1)} \\ &= \frac{F(2) - F(1)}{1 - F(1)} \\ &= \frac{\frac{1}{2} - \frac{1}{8}}{1 - \frac{1}{8}} \\ &= \boxed{\frac{3}{7}} \end{aligned}$$

Both of these parts can also be found geometrically from the graph of $f(y)$.

Example III

Let Y have density function

$$f(y) := \begin{cases} c, & 0 \leq y \leq 1, \\ 2c, & 1 < y \leq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

A. Find c.

B. Find $F(y)$, the cumulative distribution function of Y.

Example III

$$f(y) := \begin{cases} c, & 0 \leq y \leq 1, \\ 2c, & 1 < y \leq 2 \end{cases}$$

A.

$$\begin{aligned}
\int_0^2 f(y)\,dy &= c\left(\int_0^1 1\,dy + \int_1^2 2\,dy\right) = 3c = 1 \\
c &= \boxed{\frac{1}{3}}
\end{aligned}$$

B.

$$\begin{aligned}
F(y) &= \int_0^y f(t)\,dt \\
&= \begin{cases} \frac{1}{3}\int_0^y 1\,dt, & 0 \le y \le 1, \\ \frac{1}{3}\left(\int_0^1 1\,dt + \int_1^y 2\,dt\right), & 1 < y \le 2 \end{cases} \\
&= \boxed{\begin{cases} 0, & y < 0, \\ \frac{1}{3}y & 0 \le y \le 1, \\ \frac{1}{3}\left[1 + 2(y-1)\right] & 1 < y \le 2, \\ 1, & 2 < y \end{cases}}
\end{aligned}$$

As a check, we get $F(1) = 0$, $F(1) = \frac{1}{3}$ using either part of the function, and $F(2) = 1$.

Example IV

As in Example III, let Y have density function

$$f(y) := \begin{cases} \frac{1}{3}, & 0 \le y \le 1, \\ \\ \frac{2}{3}, & 1 < y \le 2. \end{cases}$$

Find $P\left(Y \leq \frac{3}{2} \middle| Y \geq \frac{1}{2}\right)$.

$$
\begin{aligned}
P\left(Y \leq \frac{3}{2} | Y \geq \frac{1}{2}\right) &= \frac{P\left(\frac{1}{2} \leq Y \leq \frac{3}{2}\right)}{P\left(Y \geq \frac{1}{2}\right)} \\
&= \frac{F\left(\frac{3}{2}\right) - F\left(\frac{1}{2}\right)}{1 - F\left(\frac{1}{2}\right)} \\
&= \frac{\frac{2}{3} - \frac{1}{6}}{1 - \frac{1}{6}} = \frac{\frac{1}{2}}{\frac{5}{6}} = \boxed{\frac{3}{5}}
\end{aligned}
$$

Example V

Let Y have cumulative distribution function

$$
F(y) := \begin{cases} 0, & y \leq 0, \\ \frac{y}{4}, & 0 < y \leq 1, \\ \frac{y^2}{4}, & 1 < y \leq 2, \\ 1, & 2 < y. \end{cases}
$$

A. Find $f(y)$, the density function of Y.

B. Find $P\left(\frac{1}{2} \leq Y \leq \frac{3}{2}\right)$.

Example V

$$F(y) := \begin{cases} \dfrac{y}{4}, & 0 < y \leq 1, \\ \\ \dfrac{y^2}{4}, & 1 < y \leq 2 \end{cases}$$

Find $f(y)$ and $P\left(\dfrac{1}{2} \leq Y \leq \dfrac{3}{2}\right)$.

A.

$$f(y) \;=\; F'(y) = \boxed{\begin{cases} 0, & y \leq 0, \\ \\ \dfrac{1}{4}, & 0 < y \leq 1, \\ \\ \dfrac{y}{2}, & 1 < y \leq 2, \\ \\ 0, & 2 < y \end{cases}}$$

B.

$$\begin{aligned} P\left(\frac{1}{2} \leq Y \leq \frac{3}{2}\right) &= F\left(\frac{3}{2}\right) - F\left(\frac{1}{2}\right) \\ &= \frac{\frac{9}{4}}{4} - \frac{\frac{1}{2}}{4} = \boxed{\frac{7}{16}} \end{aligned}$$