IV. Applications, modeling, and word problems

Lesson Overview

- **Mixing**: Smoke flows into the room; evenly mixed air flows out.

  Salt water/pollution/clean water flows into a tank/lake; evenly mixed water flows out.
- **Population**: A species reproduces and gets hunted.
- **Finance**: You make savings or pay off a loan.

Lesson Overview

1. **Set variables**:

$$\begin{aligned} t &:= \text{time} \\ y(t) &:= \text{amount of pollution/population} \\ & \text{amount of savings or debt} \end{aligned}$$

2. **Write differential equation**:

$$\begin{aligned} y'(t) &= \text{rate of change} \\ &= \underline{\text{increase}} - \underline{\text{decrease}} \end{aligned}$$

   Generally, one is constant (or a simple function of t, not of y), and the other depends on $y(t)$.

Lesson Overview

3. **Solve it**: They're usually linear, or if you're lucky, separable. Use initial data if given; otherwise use $y(0) = y_0$.

4. **Answer questions**: When does your investment double? ($y(t) = 2y_0$) When is the salt diluted to 1%? ($y(t) = \frac{1}{100}y_0$) What should your initial investment be to get 1 million in 20 years? ($y(20) = 1$ mil; solve for $y(0)$.)

Example I

We have a 100 gallon tank that is initially full of pure water. Water that is 3% salt enters at a rate of 10 gallons/minute. Mixed water leaves.

A. Write an initial value problem to describe this situation.

B. Solve it.

A.

$$\begin{aligned} t &:= \text{time} \\ y(t) &:= \text{amount of salt at time } t \\ y'(t) &= \underline{\text{increase}} - \underline{\text{decrease}} \\ &= \boxed{\frac{3}{10} - \frac{1}{10}y(t)} \end{aligned}$$

B.

$$\begin{aligned}
y' + \frac{1}{10}y &= \frac{3}{10} \\
I(t) = e^{\int \frac{dt}{10}} = e^{\frac{t}{10}} \Longrightarrow \quad e^{\frac{t}{10}}y' + \frac{1}{10}e^{\frac{t}{10}}y &= \frac{3}{10}e^{\frac{t}{10}} \\
e^{\frac{t}{10}}y &= 3e^{\frac{t}{10}} + C \\
y &= 3 + Ce^{-\frac{t}{10}} \\
y(0) = 0 \Longrightarrow \quad C &= -3 \\
y &= \boxed{3 - 3e^{-\frac{t}{10}}}
\end{aligned}$$

Example II

Consider the water tank described above.

A. How long until the tank is 1% salt?

B. What is the long-term concentration of salt?

$$\boxed{y = 3 - 3e^{-\frac{t}{10}}}$$

A.

$$\begin{aligned}
y(t) &= 1 \\
3 - 3e^{-\frac{t}{10}} &= 1 \\
3e^{-\frac{t}{10}} &= 2 \\
e^{-\frac{t}{10}} &= \frac{2}{3} \\
-\frac{t}{10} &= \ln\frac{2}{3} \\
\frac{t}{10} &= -\ln\frac{2}{3} = \ln\frac{3}{2} \\
t &= 10\ln\frac{3}{2} \\
&\approx \boxed{4.05 \text{ minutes}}
\end{aligned}$$

B. $y(\infty) = 3$, so the limiting concentration is 3 gallons out of 100, or $\boxed{3\%}$, which agrees with our intuition.

Example III

The deer population of a national park increases at a rate proportional to its current population, doubling every 69 years. There are 1,750 deer today.

A. Write an initial value problem to describe this situation.

B. Solve it.

The normal growth satisfies the following IVP:

$$\begin{aligned}
y' &= ky, y(0) = 1750 \\
\frac{dy}{dt} &= ky \\
\frac{dy}{y} &= k\,dt \\
\ln y &= kt + C \\
y &= e^{kt+C} = De^{kt} \\
y(0) = D &\implies D = 1750 \\
\\
y &= 1750e^{kt}
\end{aligned}$$

Example III

Since it doubles in 69 years, we can solve for k:

$$\begin{aligned} y(69) &= 3500 \\ 1750e^{69k} &= 3500 \\ e^{69k} &= 2 \\ 69k &= \ln 2 \approx 0.69 \\ k &= \frac{1}{100} \\ y &= \boxed{1750e^{\frac{1}{100}t}} \end{aligned}$$

Example IV

In the park above, to control the deer population, the authorities allow hunters to kill 20 deer per year. Will the population increase or decrease from then on? If it increases, when will the population reach 3,000? If it decreases, when will the population die out?

$$\begin{aligned} y'(t) &= \underline{\text{increase}} - \underline{\text{decrease}} \\ y' &= ky - 20, y(0) = 1750 \\ y' - ky &= -20 \qquad \left\{\text{Use } I(t) = e^{\int(-k)\,dt} = e^{-kt} \right\} \\ e^{-kt}y' - ke^{-kt}y &= -20e^{-kt} \\ (e^{-kt}y)' &= -20e^{-kt} \\ e^{-kt}y &= \frac{20}{k}e^{-kt} + C \qquad \left\{k = \tfrac{1}{100} \text{ from above} \right\} \\ &= 2000e^{-kt} + C \\ y &= 2000 + Ce^{kt} \\ y(0) &= 2000 + C = 1750 \\ C &= -250 \\ y &= \boxed{2000 - 250e^{kt}} \end{aligned}$$

Example IV

In the park above, to control the deer population, the authorities allow hunters to kill 20 deer per year. Will the population increase or decrease from then on? If it increases, when will the population reach 3,000? If it decreases, when will the population die out?

$$\boxed{y = 2000 - 250e^{kt}}$$

From this, we can tell that the population is $\boxed{\text{decreasing}}$. To see how long we will last, we solve $y(t) = 0$ for t:

$$\begin{aligned}
2000 - 250e^{kt} &= 0 \\
250e^{kt} &= 2000 \\
e^{kt} &= 8 \\
kt &= \ln 8 = 3\ln 2 = 3(0.69) = 2.07 \\
t &= \frac{2.07}{k} = \boxed{207 \text{ years}}
\end{aligned}$$

Example V

A student borrows \$20,000 for college at an annual interest rate of 5%. She agrees to make payments continuously at a rate of \$1,000 per month.

A. Write an initial value problem to describe this situation.

B. Assuming you could solve the equation, describe how you would determine when the loan would be fully paid off.

Example V

A student borrows \$20,000 for college at an annual interest rate of 5%. She agrees to make payments continuously at a rate of \$1,000 per month.

A.

$$\begin{aligned} t &:= \text{time, measured in months} \\ y(t) &:= \text{amount of money owed at time } t\text{, measured in dollars} \\ y'(t) &= \text{increase (from interest)} - \text{decrease (from payments)} \\ y'(t) &= \frac{0.05}{12}y(t) - 1,000, \quad y(0) = 20,000 \end{aligned}$$

B. Solve $y(t) = 0$ for t; the answer will be the number of months it will take to pay off the loan.