XVIII. Review of linear algebra

Lesson Overview

- We can multiply an $m \times n$ matrix (m rows, n columns) by an $n \times p$ matrix to get an $m \times p$ matrix:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

- You can take <u>determinants</u> as follows:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

3×3 determinants

- $$\begin{aligned} & \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \\ = \; & aei + bfg + cdh - afh - bdi - ceg \end{aligned}$$

Eigenvalues and eigenvectors

- If you have the equation

$$A\mathbf{v} = r\mathbf{v}$$

where A is an $n \times n$ matrix, $\mathbf{v}$ is an $n \times 1$ column vector, and r is a scalar, then $\mathbf{v}$ is called an <u>eigenvector</u> of A and r is called the <u>eigenvalue</u>.

How to find eigenvalues

- To find the eigenvalues of a matrix A, solve the equation

$$\det(A - rI) = 0$$

to get the characteristic polynomial in r. For each eigenvalue r that you find, solve the equation

$$(A - rI)\mathbf{v} = \mathbf{0}$$

to find the corresponding eigenvector $\mathbf{v}$.

Example I

Multiply the matrices:

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 3 & -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 5 \\ 2 & 7 & -3 \\ 3 & 1 & 2 \\ 0 & -4 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 6 & 18 & -2 \\ 2 & -30 & 25 \end{pmatrix}}$$

Example II

Write the following system of equations in matrix multiplication form:

$$\begin{array}{lcccccccccc} ① & x & & & + & 2z & & & = & 5 \\ ② & 3x & + & 2y & + & 12z & + & w & = & 16 \\ ③ & x & + & y & + & 5z & + & w & = & 5 \end{array}$$

$$\boxed{\begin{pmatrix} 1 & 0 & 2 & 0 \\ 3 & 2 & 12 & 1 \\ 1 & 1 & 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 16 \\ 5 \end{pmatrix}}$$

Example III

Write the following system of differential equations in matrix multiplication form:

$$\begin{aligned} x_1'(t) &= 2x_1(t) - x_2(t) + e^t \\ x_2'(t) &= 2x_1(t) + x_2(t) + \sin t \end{aligned}$$

$$\begin{aligned} x_1' &= 2x_1 - x_2 + e^t \\ x_2' &= 2x_1 + x_2 + \sin t \\ \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^t \\ \sin t \end{pmatrix} \\ \mathbf{x}' &= \boxed{\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ \sin t \end{pmatrix}} \end{aligned}$$

Example IV

Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 6 \end{pmatrix}$$

1. Solve $\det(A - rI) = 0$ for the eigen<u>value</u> r.
2. Solve $(A - rI)\mathbf{x} = \mathbf{0}$ for the eigen<u>vector</u> $\mathbf{x}$ corresponding to the eigenvalue r.

Example:

$$\det \begin{pmatrix} 2-r & 3 \\ -1 & 6-r \end{pmatrix} = r^2 - 8r + 15 = 0$$

Defn: $\det(A - rI)$, a polynomial in r, is the <u>characteristic polynomial</u> of A.

Solution: We get $r = 3$ <u>or</u> $r = 5$!

We already saw that an eigenvector corresponding to $r = 3$ is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Find one for $r = 5$:

$$\begin{aligned} \begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \mathbf{0} \\ &\Longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\ y &= t \\ x &= t \\ \mathbf{x} &= t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

So $\boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ is an eigenvector corresponding to $r = 5$ (and so is any nonzero multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$).

Example V

Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$\begin{aligned}
\det\begin{pmatrix} -1-r & -2 & -2 \\ 1 & 2-r & 1 \\ -1 & -1 & -r \end{pmatrix} &= (-1-r)(2-r)(-r)+2+2+(-1-r)+2(-r)-2(2-r) \\
&= -r\left(r^2-r-2\right)+4-1-r-2r-4+2r \\
&= -r^3+r^2+2r-1-r \\
&= -r^3+r^2+r-1 \\
\text{Grouping trick:} \quad &= \left(-r^3+r^2\right)+(r-1) \\
&= r^2(-r+1)+-1(-r+1) \\
&= \left(r^2-1\right)(-r+1) \\
&= (r-1)(r+1)(1-r)=0 \\
r &= \boxed{1 \text{ or } -1}
\end{aligned}$$

Example V

$$\begin{aligned}
A &= \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \\
r &= \boxed{1 \text{ or } -1}
\end{aligned}$$

$$\begin{aligned}
r=1: \quad \begin{pmatrix} -2 & -2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \mathbf{0} \\
\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \mathbf{0} \\
&\Longrightarrow \quad t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}+s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

So we have two eigenvectors, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ corresponding to the eigenvalue $r = 1$. (Actually we always have infinitely many eigenvectors, since we can always take scalar multiples of an eigenvector and get other eigenvectors. The key point here is that we have two linearly independent eigenvectors, spanning an eigenspace of dimension two corresponding to $r = 1$.)

$$r = -1: \quad \begin{pmatrix} 0 & -2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

$$\text{Switch and clear:} \quad \begin{pmatrix} 1 & 3 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

$$\Longrightarrow \boxed{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}$$