X. Inhomogeneous equations: variation of parameters

Lesson Overview

- To solve the (linear, second-order, inhomogeneous, constant coefficient) differential equation

$$y'' + by' + cy = g(t)$$

first solve the homogeneous equation

$$y'' + by' + cy = 0$$

by the methods of earlier lectures to get

$$y_{\text{hom}} = c_1 y_1 + c_2 y_2.$$

- Then find a particular solution to the inhomogenous equation

$$y'' + by' + cy = g(t)$$

using variation of parameters. This means you guess

$$y_{\text{par}} = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where $u_1(t)$ and $u_2(t)$ are solutions to the system:

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= g \end{aligned}$$

- Solve the systerm for u_1' and u_2'.
- Integrate to get u_1 and u_2.

- Plug in to $y_{\text{par}} = u_1 y_1 + u_2 y_2$.
- Hint on solving the system:
- Find the Wronskian of the two homogeneous solutions:

$$W(y_1, y_2) := \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}, |W| = y_1 y_2' - y_2 y_1'$$

- Then you can find u_1' and u_2':

$$\begin{aligned} u_1' &= -\frac{y_2 g}{|W|} \\ u_2' &= \frac{y_1 g}{|W|} \end{aligned}$$

Example I

The homogeneous differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 0$$

has general solution

$$y_{\text{hom}} = c_1 t + c_2 t e^t.$$

Find the general solution to the inhomogeneous differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 2t^3.$$

Example I

$$t^2 y'' - t(t+2)y' + (t+2)y = 2t^3, y_{\text{hom}} = c_1 t + c_2 t e^t$$

$$
\begin{aligned}
g(t) &= \frac{2t^3}{t^2} = 2t \\
|W(y_1, y_2)| &= \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2e^t + te^t - te^t = t^2e^t \\
u_1' &= -\frac{gy_2}{|W|} = \frac{-2t \cdot te^t}{t^2e^t} = -2 \\
u_1 &= -2t \\
u_2' &= \frac{gy_1}{|W|} = \frac{2t \cdot t}{t^2e^t} = 2e^{-t} \\
u_2 &= -2e^{-t} \\
y_{\text{par}} &= u_1y_1 + u_2y_2 = -2t \cdot t - 2e^{-t} \cdot te^t = -2t^2 - 2t
\end{aligned}
$$

Note that $-2t$ is a multiple of the homogeneous solution y_2, so we can omit it:

$$\boxed{y_{\text{gen}} = c_1t + c_2te^t - 2t^2}$$

Example II

For the inhomgeneous differential equation

$$y'' + y = \tan t$$

solve the corresponding homogeneous equation and find the Wronskian of the solutions.

$$
\begin{aligned}
y_{\text{hom}} &= \boxed{c_1 \sin t + c_2 \cos t} \\
y_1 = \sin t, y_2 = \cos t &\implies |W| = \boxed{-1}
\end{aligned}
$$

Note: Normally, $|W|$ will be a function of t, not a number. We just got lucky here.

Example III

Solve the inhomgeneous differential equation

$$y'' + y = \tan t.$$

$$\begin{aligned}
y_{\text{hom}} &= c_1 \sin t + c_2 \cos t \\
(\sin t)u_1' + (\cos t)u_2' &= 0 \quad \{(*) \quad \} \\
(\cos t)u_1' - (\sin t)u_2' &= \tan t \quad \{(**) \quad \} \\
y_1 = \sin t, y_2 = \cos t, g = \tan t &\Longrightarrow |W| = -1 \\
u_1' &= \frac{-y_2 g}{|W|} \\
u_2' &= \frac{y_1 g}{|W|} \\
u_1' &= \cos t \tan t = \sin t, \text{ so } u_1 = -\cos t \\
u_2' &= -\sin t \tan t = -\frac{\sin^2 t}{\cos t} = \frac{\cos^2 t - 1}{\cos t} = \cos t - \sec t \\
u_2 &= \sin t - \ln(\sec t + \tan t) \\
y_{\text{gen}} &= c_1 \sin t + c_2 \cos t - \sin t \cos t + \sin t \cos t - \cos t \ln(\sec t + \tan \\
&= \boxed{c_1 \sin t + c_2 \cos t - \cos t \ln(\sec t + \tan t)}
\end{aligned}$$

Example IV

For the inhomgeneous differential equation

$$y'' + 3y' + 2y = \sin e^t$$

solve the corresponding homogeneous equation and find the Wronskian of the solutions.

$r = -1, -2 \Longrightarrow \boxed{y_{\text{hom}} = c_1 e^{-t} + c_2 e^{-2t}}$

$W(y_1, y_2) := \begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix}, |W| = -2e^{-3t} + e^{-3t} = \boxed{-e^{-3t}}$

Example V

Solve the inhomgeneous differential equation

$$y'' + 3y' + 2y = \sin e^t.$$

$$\begin{aligned}
u_1' &= -\frac{y_2 g}{|W|} = \frac{-e^{-2t}\sin e^t}{-e^{-3t}} = e^t \sin e^t && \left\{\text{Integrate using } s = e^t, ds = e^t\,dt.\right\} \\
u_1 &= -\cos e^t \\
u_2' &= \frac{y_1 g}{|W|} = \frac{e^{-t}\sin e^t}{-e^{-3t}} = -e^{2t}\sin e^t && \left\{\text{Integrate using } s = e^t, ds = e^t\,dt.\right\} \\
-\int s \sin s\, ds &= -(-s\cos s + \sin s) && \{\text{No } +C \text{ necessary.}\} \\
u_2 &= e^t \cos e^t - \sin e^t
\end{aligned}$$

Plug in:

$$\begin{aligned}
y_{\text{par}} &= -e^{-t}\cos e^t + e^{-t}\cos e^t - e^{-2t}\sin e^t \\
&= -e^{-2t}\sin e^t \\
y_{\text{gen}} &= \boxed{c_1 e^{-t} + c_2 e^{-2t} - e^{-2t}\sin e^t}
\end{aligned}$$